Statistical Self-Similarity: Fractional Brownian Motion

SIKAZWE, Geofrey Wingi

Coordinator: Tuomo Karaunne

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Abstract

The objective of this paper is to provide an introductory review on fractional Brownian motion (fbm), the area which is relatively new. The basic statistical property of self-similarity and its relation to fractional Brownian motion is well discussed. Specific emphasis is given to the relation between the properties of fbm and modelling financial markets.

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1. Introduction

1.1 Hurst's Statistical Phenomenon

1.1.1 Background in brief

The concept of Hurst's statistical phenomenon have long history. It was introduced by the British Hydrologist Harold Edwin Hurst in 1951. Hurst spent more than 60 years in Egypt as a participant of a Nile hydrology project.

The Hurst's statistical phenomenon is essentially the tendency of wet years to cluster into wet periods or of dry years to cluster into drought periods. An alternative term associated to Hurst's statistical phenomenon is called 'Joseph effect' introduced by Mendelbrolt (1977).

1.1.2 Discovery of Hurst's parameter or exponent

Suppose that $X_i : i = 1, 2, \ldots, n$ represents the values of n successive yearly water run-offs (of Nile river). Define the estimate for the expectation X_k as $\bar{X}_k = \frac{1}{n}\sum_{k=1}^{n} X_k$. The deviation of the cumulative value X_k corresponding to k successive years from the mean is given by,

$$\text{Mean deviation} = X_k - \bar{X}_k.$$

The minimum and muximum deviations are defined by,

$$\min_{k \leq n}\left(X_k - \bar{X}_k\right) \quad \text{and} \quad \max_{k \leq n}\left(X_k - \bar{X}_k\right). \tag{1.1.1}$$

The '*range*' characterizing the amplitude of the deviation of the cumulative values of X_k from the mean $\bar{X}_k$ over n successive years is given by,

$$R_n = \max_{k \leq n}\left(X_k - \bar{X}_k\right) - \min_{k \leq n}\left(X_k - \bar{X}_k\right). \tag{1.1.2}$$

Let Q_n be a normalizing constant defined by,

$$Q_n = \mathcal{R}_n / \mathcal{S}_n \tag{1.1.3}$$

where,

$$\mathcal{S}_n = \sqrt{\frac{1}{n}\sum_{k=1}^{n} X_k^2 - \left(\frac{1}{n}\sum_{k=1}^{n} X_k\right)^2}. \tag{1.1.4}$$

is the empirical mean deviation. Hurst observed that, for large n, the statistic $\mathcal{R}_n/\mathcal{S}_n$ is approximately,

$$\mathcal{R}_n/\mathcal{S}_n \approx cn^{\mathbb{H}},$$

where c is a certain constant and the parameter $\mathbb{H}$, which is now called the Hurst parameter or the Hurst exponent is approximately equal to 0.7.

1.2 Statistical Self-Similarity

Definition 1.2.1 ([2]). *Let T be either $\mathbb{R}, \mathbb{R}_+ = \{t : t \geq 0\}$ or $\{t : t > 0\}$. The real-valued process $\{X(t), t \in T\}$ is self-similar with index $H > 0 \quad (H - ss)$ if for all $a > 0$, the finite-dimensional distribution of $\{X(at), t \in T\}$ are identical to the finite-dimensional distribution of $\left\{a^H X(t), t \in T\right\}$; i.e., if $d \geq 1, t_1, t_2, \ldots, t_d \in T$ and any $a > 0$,*

$$(X(at_1), \ldots, X(at_d)) \stackrel{d}{=} \left(a^H X(t_1), \ldots, a^H X(t_d)\right). \tag{1.2.1}$$

Remark 1.2.2. *The relation in (1.2.1) can be expressed as;*

$$\{X(at), t \in T\} \stackrel{d}{=} \left\{a^H X(t), t \in T\right\}.$$

For fixed $t \in T$, $X(at) \stackrel{d}{=} a^H X(t)$ means that the random variables $X(at)$ and $a^H X(t)$ have identical distributions.

Proposition 1.2.3 ([3]). *If $X(t), 0 < t < \infty$ is H-ss, then;*

$$Y(t) = e^{tH} X(e^t), -\infty < t < \infty \quad \textit{is stationary.} \tag{1.2.2}$$

Conversely, if $Y(t), -\infty < t < \infty$ is stationary, then;

$$X(t) = t^H Y(\ln t), 0 < t < \infty \quad \textit{is H-ss.} \tag{1.2.3}$$

A classical example of a self-similar process is a Brownian motion $X = X(t)t \geq 0$. Recalling that for a Gaussian process: $EX(t) = 0, EX_s X_t = min(s, t)$. Hence, $EX_{as}X_{at} = min(as, at) = amin(s, t) = E(a^{\frac{1}{2}}X_s)(a^{\frac{1}{2}}X_t)$, and the 2-d distributions $\text{Law}(X_s, X_t)$ have the property of statistical self-similarity with Hurst exponent $H = \frac{1}{2}$. The process of this example have independent increments.

Remark 1.2.4. *$Law(X_{at}) = Law(X_{bt})$, means that X_{at} and X_{bt} have the same joint distribution.*

2. Fractional Brownian Motion (FBM)

2.1 Background

Fractional Brownian Motion (FBM) belongs to a class of long-memory Gaussian process that can be represented as linear functionals of an infinite dimensional Markov process. FBM is a generalization of the more well-known process of Brownian motion. It is a centred Gaussian process with stationary increments. However, the increments of the fractional Brownian motion are not independent, except in the standard Brownian case. The dependence structure of the increments is modeled by a parameter $H \in (0, 1)$.

The name fractional Brownian motion comes from the influential paper by Mandelbrot and Van Ness [4]. They defined the fractional Brownian motion as a fractional integral with respect to the standard one (whence the name). The notation for the index H and the current parametrization with range (0,1) are due to Mandelbrot and Van Ness also. The parameter H is called the Hurst index after an English hydrologist who studied the memory of Nile River maxima in connection of designing water reservoirs [6].

2.2 Model Definition

Consider the function,

$$B(s,t) = |s|^{2H} + |t|^{2H} - |t-s|^{2H}, \quad s,t \in \mathbb{R}. \tag{2.2.1}$$

We note that the function in equation (2.2.1) is a nonnegative definite. Therefore, there exists a Gaussian process on some probability space (e.g. on a real function $w = (w_t), t \in \mathbb{R}$), that has a zero mean and the auto covariance function,

$$Cov(B_s, B_t) = \frac{1}{2} B(s,t),$$

i.e. the process such that,

$$EB_sB_t = \frac{1}{2}\left\{|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right\}. \tag{2.2.2}$$

Hence,

$$EB_{as}B_{at} = a^{2H} EB_sB_t = E\left(a^H B_s\right)\left(a^H B_t\right).$$

so that,

$$\mathrm{Law}(B_{as}, B_{at}) = \mathrm{Law}\left(a^H B_s, a^H B_t\right). \tag{2.2.3}$$

In other words, the random function of (B_s, B_t) have the same joint distribution with $\left(a^H B_s, a^H B_t\right)$. We conclude from this observation that B is a self-similar process with Hurst exponent H. It follows from equation (2.2.1) that,

$$E|B_t - B_s|^2 = |t-s|^{2H}. \tag{2.2.4}$$

Definition 2.2.1 ([7]). *A standard Fractional Brownian Motion with Hurst self-similar exponent $0 < H \leq 1$ is a continuous Gaussian process $B = (B_t) \geq 0$ with zero mean and a covariance function given in* (2.2.1).

We denote the Fractional Brownian Motion (FBM) by $B_H = (B_H(t))t \geq 0$. From the definition 2.2.1 we note the following:

(a) $B_H(0) = 0$,

(b) $EB_H(t) = 0$ and,

(c) $EB_H(t)B_H(s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), s, t \in \mathbb{R}$.

2.3 Properties of FBM

(a) $B_H(0) = 0$ and $EB_H(t) = 0$ for all $t > 0$, means that the process has stationary increments.

(b) Hurst self-similar; e.g. $B_H(t) \sim t^H B_H(t)$. There exists various generalization of self-similarity property. For example: let $X(\alpha) = (X_t(\alpha))t \geq 0$ be a process of *Ornstein-Uhlenbeck* type with parameter $\alpha \in \mathbb{R}$, i.e, a Gaussian Markov process defined by the formula,

$$X_t(\alpha) = \int_0^t e^{\alpha(t-s)} dW_s, \quad t \geq 0. \tag{2.3.1}$$

where $W = (W_s)s \geq 0$ is a standard Gaussian motion.

Note 2.3.1. *$X(\alpha) = (X_t(\alpha))t \geq 0$ is a solution of a the stochastic linear differential equation, $dX_t(\alpha) = \alpha X_t(\alpha)dt + dW_t, \quad X_0(0) = 0$. From* (2.3.1)*, we observe that*

$$Law\left(X_{at}(\alpha), t \in \mathbb{R}\right) = Law\left(a^{\frac{1}{2}} X_t(\alpha) \in \mathbb{R}\right).$$

for $a \in \mathbb{R}$ it is regarded as a self-similar process.

(c) Anti persistent. The increments in Figure 2.1 are negatively correlated and the process is said to have short-range dependence. Short-range dependence is a measure of the decline in statistical dependence of two events separated by successively longer spans of time. Heuristically, a time series is strong mixing if the maximal dependence between any two events becomes trivial as more time elapses between them.

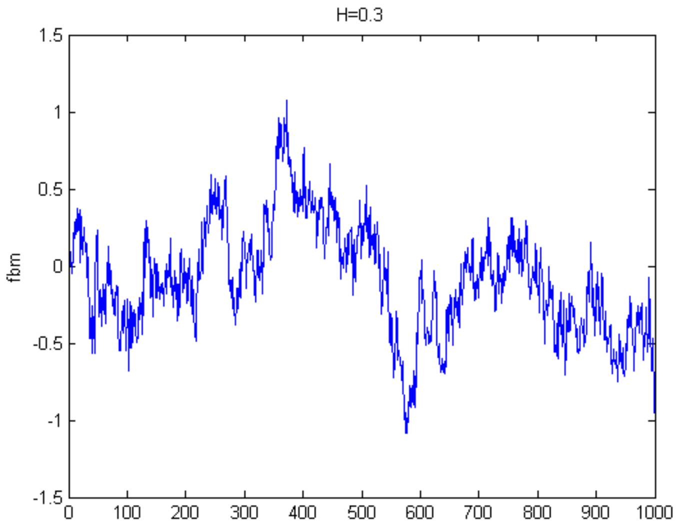

Figure 2.1: The Figure for $0 < H < \frac{1}{2}$: The increments are negatively correlated.

(d) Independent increments.

(e) Persistent. Figure 2.3 show that the increments are positively correlated and the process is said to have long-range dependence. This means that the auto covariance function decays so slowly as time tends to infinity.

2.4 Rationale of FBM in Finance

- H-ss, where H takes values between 0 and 1, makes FBM more flexible as a modelling tool than standard Brownian motion that only allows $\frac{1}{2} - \mathrm{ss}$.
- The existence of long-range dependence and positive correlation of future and past increments makes FBM especially an attractive pricing tool.

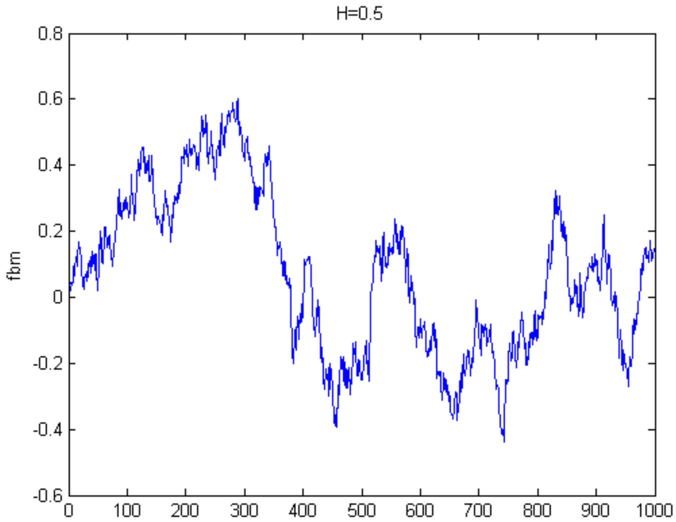

Figure 2.2: Figure for $H = \frac{1}{2}$: The increments are independent

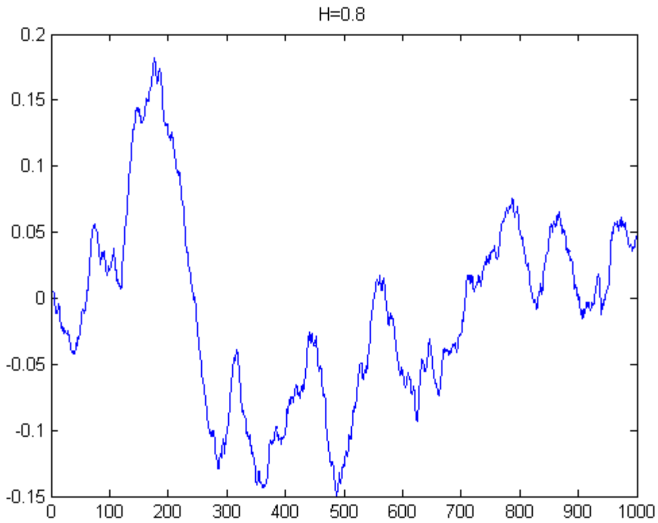

Figure 2.3: The Figure for $\frac{1}{2} < H < 1$: The motion have long-range memory

- Statistical analysis has indicated that most markets are monofractal, have Hurst parameters between 0.5 and 1. Thus, for financial purposes, Hurst parameter are usually assumed to be between 1/2 and 1 exclussively.

3. Applications of FBM

3.1 Option Pricing and Volatility Estimation by FBM

The reference is made to a recent study by Cajueiro and Barbachan (2003) [1]. The main objectives was to study the behaviour of Brazilian stock returns in continuous time. The focus was on the dependency in Brazilian stock returns increments. They also shown how to price European options using Black-Schole type formula derived for Fractional Brownian Motion.

Most of the previous studies on stock returns concentrated on fat tail issue. A typical example is a GARCH model which have been specifically used in discrete time.

3.1.1 The Market Model

The model under consideration here is the fractional Black-Scholes market. Under the FBS market model, the investor is confronted with two investment possibilities:

(a) Investment in a risky-free asset whose price is given by;

$$dB(t) = \rho B(t)dt \quad [B(0) = 1, 0 < t \leq T, \rho > 0]. \tag{3.1.1}$$

(b) Investment in a risky-asset whose price satisfies:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t) \quad [S(0) = s_0], \tag{3.1.2}$$

where, ρ, μ and σ are constants, B_H is a fractional Brownian motion with a Hurst's parameter H.

It has been shown that fractional Black-Scholes market presents no arbitrage opportunity and is thus complete. The solution to the equation (3.1.2) is given by;

$$S(t) = s_0 \exp\left(\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right). \tag{3.1.3}$$

3.1.2 Option Pricing Formula

We consider the fractional Black-Scholes formula for option pricing. The price at every time $t \in [0, T]$ of a European call option with strike price K and maturity time T is given by;

$$C = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2), \tag{3.1.4}$$

where,

$$d_1 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) + \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}.$$

and,

$$d_2 = \frac{\ln\left(\frac{S(t)}{K}\right) + r(T-t) - \frac{\sigma^2}{2}\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}}.$$

3.1.3 Data and Results

In Figure 3.1 the plot FBS - BS call option $\times$ Maturity $\times$ Exercise price is shown. One can observe that for $H = 0.59$, the price given by the FBS-Black- Scholes formula is bigger than the price given by the ordinary Black-Scholes formula (it happens for any $H > \frac{1}{2}$).

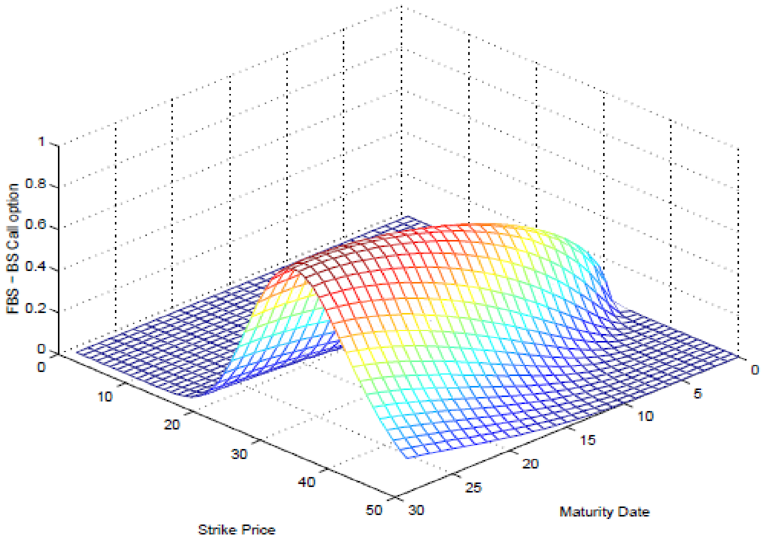

Figure 3.1: FBS-Maturity-Exercise Price

4. Discussion

In this paper, we studied the concept of statistical self similarity with a special emphasis given to fractional Brownian motion. The basic properties of the fractional Brownian motion were clearly reviewed as shown in figures 2.1, 2.2 and 2.3.

The Hurst's parameter H determines the sign of the covariance of the future and past increments. This covariance is positive when $H > \frac{1}{2}$ which means that the process is persistent, zero when $H = \frac{1}{2}$ and negative when $H < \frac{1}{2}$ which means that the process is anti-persistent. In fact, these properties are remarkable features of the fractional Brownian motion. When one says that the process is persistent, it means that, if the process had a positive increment in the past, i.e. an increase – then it will also have on the average an increase in the future. Therefore, an increase trend in the past implies an increasing trend in the future. On the other hand, if the process is antipersistent, then an decreasing trend in the past implies an increasing trend in the future.

Fractional Brownian motion have been proved to be an important tool in modelling financial markets. In this paper, we reviewed how fractional Brownian motion is used in option pricing and volatility Estimation. In particular, the behaviour of Brazilian stock returns in continuous time, were studied with emphasis to the dependency in Brazilian stock returns increments.

It is believed that the FBM model does not deal with the fat tail behavior observed in stock returns, so it would be interesting to consider models with dependent increments and fat tailed returns, especially to understand better the Brazilian stock returns since they present both fat tails and serial dependency.

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