Some aspects of Ito and Stratonovich integral in Stochastic Differential Equations(SDE)

Presenter: Abola Benard
Laboratory of Applied Mathematics
Lappeenranta University of Technology

January 10, 2012

OUTLINES

(1) Some basic definitions and concepts.

(2) Ito's and Stratonovich integrals, and compare them.

(3) Applications of the integrals in SDE.

(4) Conclusion.

1) SOME BASICS DEFINITIONS AND CONCEPTS

1. [Brownian motion] (Wiener process). The stochastic process $\mathbf{W} = \{W_t : t > 0\}$ is called a standard Bromnian motion if:-

   - $W_0 = 0$
   - $\mathbf{W}$ has independent and stationary increments.
   - W_t is $N(0, t)$.
   - $W_t - W_s$ is $N(0, t - s)$ for $s < t$. (i.e. stationary independent increments)

1) SOME BASICS DEFINITIONS AND CONCEPTS (continous)

2. [Martingale] A process $\mathbf{S} = \{S_t : t > 0\}$ is a martingale with respect to information set I_t and with respect to probability density $P(.)$ if:-

   - S_t is known, given I_t
   - $\mathbf{E}(S_t) < \infty$
   - $\mathbf{E}(S_t) = S_t$ and $\mathbf{E}(S_{t+1}|I_t) = S_t$.

   In other words, the process S_t is martingale if its future movement is unpredictable given the information at hand.

1) SOME BASICS DEFINITIONS AND CONCEPTS (continous)

3. [Properties of Martingale] Let $\Delta S_t = S_{t+1} - S_t$ be martingale difference, then:-

   - $\sum_{t=0}^{n} |\Delta S_t|$ is called the length of trajectory.
   - $\sum_{t=0}^{n} (\Delta S_t)^2$ is known as quadratic variation [very important!!].
   - $\sum_{t=0}^{n} |(\Delta S_t)|^3$ and $\sum_{t=0}^{n} (\Delta S_t)^4$ exist and they important too.

1) SOME BASICS DEFINITIONS AND CONCEPTS (continous)

4. [Example on Martingale]
   Let S_t be the price of an asset observed at time t and if in a small infinitesimal period δt, one receives a new information on S_t, denoted by dS_t. This change can be expressed as

   - $dS_t = \sigma dW(t)$, where $W(t)$ is Wiener process and σ is the volatility of the asset.

   The expectation, $\mathbf{E} dS_t = 0$, this demonstrates how Wiener process and martingale are related.

1) SOME BASICS DEFINITIONS AND CONCEPTS (continous)

5. [Stochastic Differential Equations (SDE)]
   A differrential equation of the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t \quad (1)$$

   is an SDE if $dW(t)$ is a wiener process (sometimes called innovation term). $\mu(X_t, t)$ and $\sigma(X_t, t)$ is drift and volatility (diffusion) coefficients respectively. The integral of Equation (1) is not trivial!!!

2) ITO's INTEGRAL
continous

The Ito isometry

$$E\left(\int_0^T f(t,w)dW\right)^2 = E\left(\int_0^T f^2(t,w)dt\right) \tag{2}$$

The consequence of this can be used to show that $\int_0^T WdW = \frac{1}{2}W^2 - \frac{T}{2}$. Which is an Ito integral of $\int_0^T WdW$, where W is Bromnian process (Wiener process)

2) ITO's INTEGRAL
continous

The Ito formula

Let X_t be an Ito process given by $dX_t = \mu dt + \sigma dW_t$. Let Y be a function of t and X_t, defined as $Y = y(t, X_t)$. Suppose its twice differentiable on $[0, \infty)$, then

$$dY_t = \frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}dt \tag{3}$$

Example: Given $y = \frac{1}{2}x^2$, then $\frac{\partial y}{\partial t} = 0$, $\frac{\partial y}{\partial x} = x$ and $\frac{\partial^2 y}{\partial x^2} = 1$. Yields a new Ito DE

$$dY = (x\mu + \frac{1}{2})dt + \sigma dW.$$

Suppose $\mu = 0$ and $\sigma = W$, then $dY = \frac{1}{2}dt + WdW$.

2) ITO's INTEGRAL
continous

Integration by parts in Ito integral

Suppose f only depend on $s \in [0,t]$ and that f is continuous and of bounded variation in $[0,t]$. Then

$$\int_0^t f(s)dW = f(t)W_t - \int_0^t W df \quad (4)$$

Example: Let X_t, Y_t be Ito processes in $\mathbf{R}$. Prove that

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + dX_t dY_t. \quad (5)$$

2) ITO's INTEGRAL
continous

Deduce the following general integration by parts formula

$$\int_0^t X_s dY_s = X_t Y_t - X_0 Y_0 - \int_0^t Y_s dX_s - \int_0^t dX_s dY_s. \qquad (6)$$

This can be done using Ito formula, $i.e$, Let $f = XY$, then

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX_t + \frac{\partial f}{\partial Y}dY_t + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}(dX_t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial Y^2}(dY_t)^2 + \frac{\partial^2 f}{\partial X \partial Y}dX_t dY_t$$

but the 1^{st}, 4^{th} and 5^{th} right terms equal to zero, and $\frac{\partial f}{\partial Y} = X_t$, $\frac{\partial f}{\partial X} = Y_t$, $\frac{\partial^2 f}{\partial X \partial Y} = 1$ and $df = d(X_t Y_t)$. Substituting in Ito formula end the proof. The last part is rearrangement and integrating.

2) STRATONOVICH INTEGRAL

Suppose f is continuous in $[0, T]$, the stratnonvich integral of f is defined as

$$\int_0^T f(t,w)dW = \lim_{\Delta t_j \to 0} \sum_j f(t^*, w)\Delta W_j, \tag{7}$$

where $t^* = \frac{1}{2}(t_{j+1} + t_j)$ and $\Delta W_j = W_{j+1} - W_j$. However, Ito integral takes $t^* = t_j$.

$$\text{Stratonovich:} \quad \int_0^t W dW = \frac{1}{2}W_t^2 \tag{8}$$

$$\text{Ito:} \quad \int_0^t W dW = \frac{1}{2}W_t^2 - \frac{1}{2}t \tag{9}$$

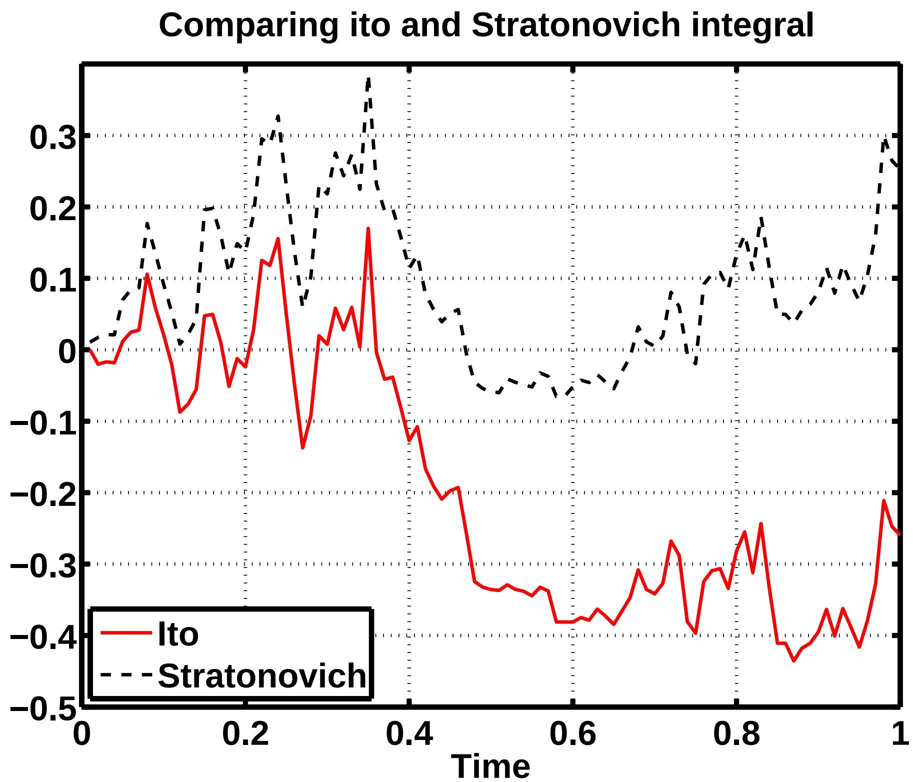

Figure 1: At all time point Stratonovich integral (equation (10))is greater than Ito integral (equation (11)) valve by half that time.

Is stratnonvich integral better Ito integral?

- Some scholars noted that Stratonovich interpretation in some situations may be most appropriate. This means that it depends on situation(s) at hand.
- If white noise is approximation to continuously fluctuating noise with finite memory (much shorter than dynamical timescales), appropriate representation is Stratonovich (Wong and Zakai, 1969)
- If white noise approximates set of discrete pulses with finite separation to which system responds, or SDE continuous approximation to discrete system, then Ito representation appropriate.
- NB: In Stratonovich DE classical differential calculus can be applied but not in Ito DE.

Transformation from Ito to stratnonvich DE .

Let a physicsl system be defined by Ito SDEs

$$dX_t \;=\; \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \tag{10}$$

then the same process can be described also with Stratonovich equation

$$dX_t \;=\; \left(\mu(X_t, t) - \frac{1}{2}\sigma'(X_t, t)\sigma(X_t, t)\right) dt + \sigma(X_t, t)dW_t \tag{11}$$

where $\sigma'(X_t, t) = \frac{\partial \sigma(X_t, t)}{\partial x}$. But if Equation (10) define Stratonovich SDE, then, Ito SDE becomes

$$dX_t \;=\; \left(\mu(X_t, t) + \frac{1}{2}\sigma'(X_t, t)\sigma(X_t, t)\right) dt + \sigma(X_t, t)dW_t \tag{12}$$

Example: Geometric SDE

Let Ito DE be defined by

$$dX_t \quad = \quad rX_t dt + \alpha X_t dW_t, \tag{13}$$

Then, we can find Stratonovich DE using Equation (11). Let $\mu = rX_t$ and $\sigma = \alpha X_t$, this implies that $\sigma' = \alpha$. Substituting in Equation (11) yields $dX_t = X_t \left[(r - \frac{1}{2}\alpha^2)dt + rdW_t\right]$.
Now the solution to the SDE can be found easily by classical DE, $i.e$, separating the variables and integrating normally.
We get

$$X_t = X_0 \exp\left[(r - \frac{1}{2}\alpha^2)t + rW_t\right].$$

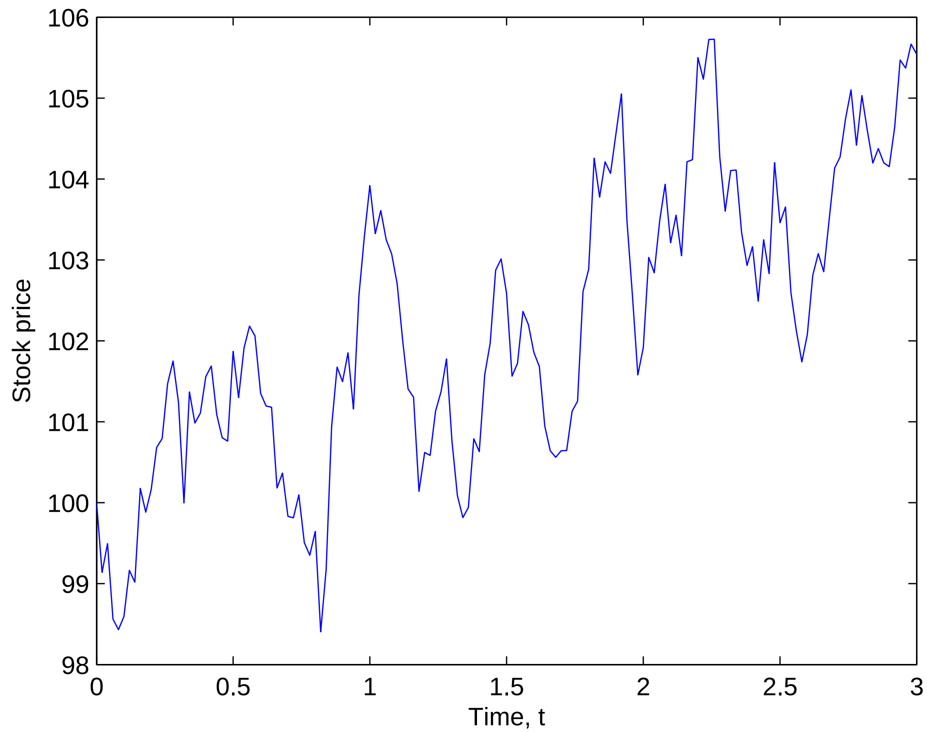

Figure 2: Geometric Brownian motion. Where $X_0 = 100$, r=0.025 $\alpha = 0.05$ and W is random white noise, $N(0, \frac{1}{50})$

Conclusion

This presentation have not tackle the multivariate aspect of Ito process in SDE.

THANKYOU FOR ATTENTION