# Ornstein-Uhlenbeck Theory

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# Definition of a stochastic process

- Let $(\Omega, \mathcal{F}, P)$ be a probability space.
- A stochastic process is a collection of random variables $X(t, \omega)$, where $\omega \in \Omega$ and $t$ is a time.
- Sometimes a stochastic process is simply written $X(t)$.

# Stationary stochastic process

- A stochastic process $X(t)$ such that $\mathbb{E}|X(t)|^2 < \infty$ is said to be stationary if for all $t_1, t_2, \ldots, t_n$ and $h > 0$, the n-random vectors $(X(t_1), X(t_2), \ldots, X(t_n))$ and $(X(t_1 + h), X(t_2 + h), \ldots, X(t_n + h))$ are identically distributed: i,e, time shifts leave joint probabilities unchanged.

# Markovian, Gaussian stochastic process

The stochastic process $X(t)$, $t \geq 0$

- is Markovian if for all $t_1 < t_2 < ... < t_n$,

  $$P(X(t_n) < x | X(t_1), X(t_2), \ldots, X(t_{n-1})) = P(X(t_n) < x | X(t_{n-1})),$$

  that is, the future is determined only by the present and not the past.
- is Gaussian if for all $t_1 < t_2 < ... < t_n$, the n-vector $(X(t_1), X(t_2), \ldots, X(t_n))$ is multivariate normally distributed.

# Stochastic process with independent increments

The stochastic process $X(t)$, $t \geq 0$

- is said to have independent increments if, for all $t_0 < t_1 < ... < t_n$, the $n$ random variables $X(t_1) - X(t_0), X(t_2) - X(t_1), ..., X(t_n) - X(t_{n-1})$ are independent.
- **Note!:** This condition implies that $X(t)$, $t \geq 0$ is Markovian, but not conversely.
- The increments are further said to be stationary if, for any $t > s$ and $h > 0$, the distribution of $X(t+h) - X(s+h)$ is the same as the distribution of $X(t) - X(s)$

# Wiener-Lévy process or Brownian motion

A stochastic process $W(t), t > 0$ is a Wiener-Lévy process or Brownian motion if

- it has stationary independent increments,
- $W(t)$ is normally distributed, and
- $E[W(t)] = 0$ for each $t > 0$, and $W(0) = 0$.

# Ornstein-Uhlenbeck process

- A stochastic process $X(t)$, $t > 0$ is continuous in probability if, for all $u \in \mathbb{R}^+$ and $\varepsilon > 0$,

$$P(|X(v) - X(u)| \geq \varepsilon) \to 0, \text{ as } v \to u.$$

- A stochastic process $X(t)$, $t > 0$ is an **Ornstein-Uhlenbeck process** or a **Gauss-Markov process** if it is stationary, Gaussian, Markovian, and continuous in probability.

# Associated SDE

- A fundamental result, due to Doob , is that the Ornstein-Uhlenbeck process necessarily satisfies the following linear stochastic differential equation:

$$dX(t) = -\rho(X(t) - \mu)dt + \sigma dW(t),$$

  where $\{W(t), t \geq 0\}$ is Brownian motion with unit variance parameter and $\mu, \rho, \sigma \geq 0$ are constants.
- The above SDE is often taken as the primary definition of an OrnsteinUhlenbeck process.

# Exact solution of Ornstein-Uhlenbeck SDE

- The SDE $dX(t) = -\rho(X(t) - \mu)dt + \sigma dW(t)$, is transformed via Itô-formula with the auxiliary function $f(x,t) = xe^{\rho t}$ to the SDE

$$d(X(t)e^{\rho t}) = e^{\rho t}\rho\mu\, dt + \sigma e^{\rho t}\, dW(t).$$

- Integrating from 0 to $t$ we get

$$X(t)e^{\rho t} = X(0) + \int_0^t e^{\rho s}\rho\mu\, ds + \int_0^t \sigma e^{\rho s}\, dW_s$$

which gives

$$X(t) = X(0)e^{-\rho t} + \mu(1 - e^{-\rho t}) + \sigma \int_0^t e^{-\rho(t-s)}\, dW_s.$$

# Simulation of Ornstein-Uhlenbeck SDE

The following graph is a single sample simulation of the SDE

$$dX = -\rho X dt + \sigma dW(t), X(0) = x_0.$$

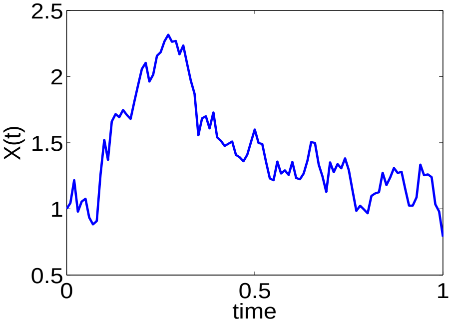


Figure: A sample path of O-U process with parameters $\mu = 0, \rho = 1, \sigma = 1$ and $x_0 = 1$

# Application of Ornstein-Uhlenbeck SDE

Ornstein-Uhlenbeck SDE has many application in

- financial mathematics : OU process is one of several approaches used to model interest rates, currency exchange rates, and commodity prices stochastically.
- physical sciences

# Application of Ornstein-Uhlenbeck SDE

Murakoze!
Kiitos!