# An interacting particle system modelling aggregation behavior: from individuals to populations.

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# Outline

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3. The Eulerian description
4. Asymptotic behavior of the system for large populations

# Introduction

- In biology and medicine there is a wide spectrum of examples which exhibit collective behavior, such as formation

  of patterns and clustering. This may happen at any scale from:

-the cellular scale

-the microscopic scale and

-the scale of animal grouping.

- The interest here is to understand the mechanism of formation of aggregates

# Cont'd

- **Mathematicaly**: Aggregation patterns are usually explained in terms of **forces, external** and/or **internal**, acting upon individuals.
- When doing the mathematical modelling The aim of the modelling is to catch the main features of the interaction at the lower scale of single individuals that are responsible, at a larger scale, for a more complex behavior that leads to the formation of aggregating patterns.
- A classical approach has been based on PDE's (**Eulerian Models**)
- These models describe the evolution of population densities, they are based on continuum equations, typically (deterministic) nonlinear

  partial differential equations of the advection-reaction-diffusion type

$$\rho_t + \nabla \cdot (v\rho) = \nabla \cdot (D \nabla \rho) + \nu(\rho),$$

# Cont’d

The advantages of this approach are:

- ease analysis and elimination of arbitrary spatial discretization;
- useful in the case of large and dense populations

The disadvantages

- the identity of individuals is compromised.
- some important features of the dynamics of the population may be hidden

# Cont'd

- A fruitful approach is based on the modelling of the "movement" of each individual in the total population of $N$ similar particles

  ( **individual based model** - IBM)

  this is called **the Lagrangian approach:**individuals are followed in their motion.

  Possible randomness may be included in the motion of a particle, so that the variation in time of the (random) location of the $k$-th individual in the group at time $t$ , $X^k_N(t)$ is described by a system of stochastic differential equations.

- Interaction may also lead to density dependence in the diffusion terms i.e from a Lagrangian point of view that,

  the state of a system of $N$ particles may be described as a (stochastic) process $\{X^k_N(t)\}$

  The evolution of each $X^k_N(t)$ will be subject to a SDE of the type:

$$dX^k_N(t) = \left[ f^k_N(t) + h^k_N(X^1_N(t), \ldots, X^N_N(t), t) \right] dt$$
$$+ \sigma \left( X^1_N(t), \ldots, X^N_N(t), t \right) dW^k(t), \quad k = 1, \ldots, N,$$

# Cont'd

- Stochastic Lagrangian models offer the advantage of being directly related to experimental data on the behavior of individuals of a real population, especially when dealing with a relatively "small" number of individuals per unit space.
- By an Eulerian approach, the collective behavior of the discrete (in the number of particles) system, may be given in terms of the spatial distribution of particles at time *t*, expressed in term of an empirical measure

$$X_N(t) = \frac{1}{N}\sum_{k=1}^{N} \epsilon_{X_N^k(t)} \in \mathcal{M}(\mathbb{R}^d),$$

such that $\forall B \in \mathcal{B}(\mathbb{R}^d)$,

$$(X_N(t))(B) = \frac{1}{N}\sum_{k=1}^{N} \epsilon_{X_N^k(t)}(B) = \frac{\sharp\text{particles in } B \text{ at time } t}{N}.$$

# Cont’d

In conclusion, the two different approaches (Lagrangian and Eulerian) describe the system at different scales:

- the finer scale description based on the (stochastic) behavior of individuals (**microscale**)
- and the larger scale description based on the (continuum) behavior of population densities (**macroscale**) .
- “The central problem is to determine how information is transferred across scales, and what detail at fine scales is exactly necessary and sufficient for understanding patterns on averaged scales”.

The aim in this analysis is to provide a mathematically rigorous framework for bridging the gap between the two different scales: the microscale and the macroscale. In order to consider a rigorous limit, they introduce in the model the concept of “**mesoscale**” .

The mesoscale is obtained by a suitable rescaling of the kernel modelling the interaction among particles.

# Cont'd

- The three types of interaction may be obtained in terms of an appropriate rescaling of a given reference function $V1$. Let $V1$ be a sufficiently regular probability density; and assume that the interaction of two particles, out of $N$, located in $x$ and $y$ respectively is modelled by

$$\frac{1}{N} V_N(x-y),$$

where

$$V_N(z) = N^{\beta} V_1(N^{\beta/d} z),$$

# Cont’d

- The force exerted on the *k*-th (out of *N*) single particle located at $X^k_N(t)$ due to the interaction of the single *k*-particle with all the others in the population is given by

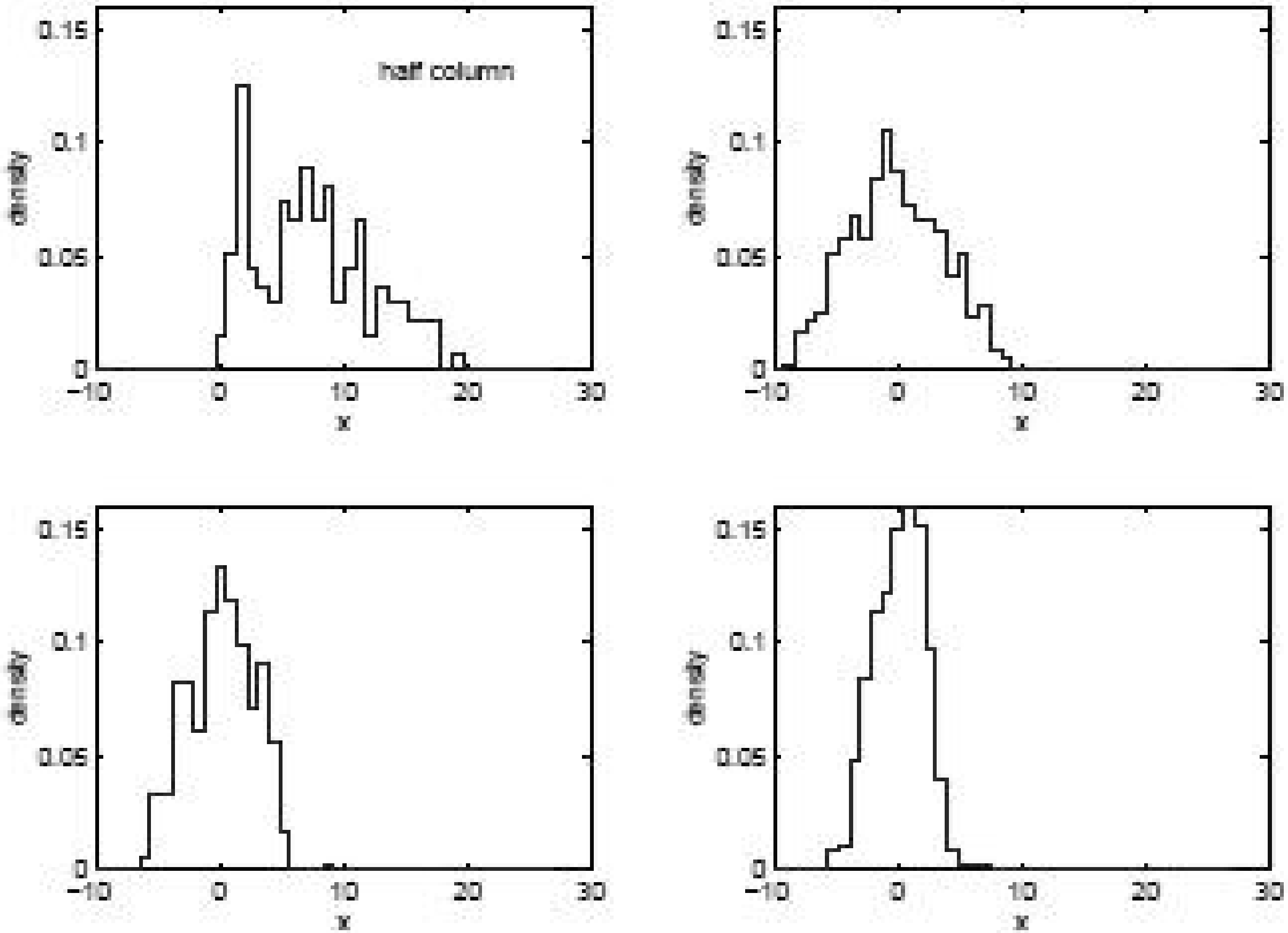
$$
\begin{aligned}
I^k \equiv I^k(X^1_N(t),\ldots,X^N_N(t)) &= \sum_{i=1}^{N} \frac{1}{N} V_N\left(X^k_N(t) - X^i_N(t)\right) \\
&= \sum_{i=1}^{N} N^{\beta-1} V_1\left(N^{\beta/d}\left(X^k_N(t) - X^i_N(t)\right)\right)
\end{aligned}
$$

$$
I^k = (X_N(t) * V_N)(X^k_N(t))
$$

# Cont'd

- The biological motivation of the present paper arises from an example of animal swarming.
- Experiments have been performed in Piemonte, on a population of ants of the species **Polyergus rufescens**.
- The colonies of these species are characterized by the absence of **polyethism** in the **worker** cast which is composed only by **soldiers**, unable to attend any task (e.g. brood tending or nest maintenance) other than raiding activity [9].
- These indispensable tasks are performed by individuals belonging to few specific species which have been kidnapped by *Polyergus* soldiers when they were newborn or pupae, and grew up in the *Polyergus* nest.
- To keep constant the slave's population in their nest, *Polyergus* ants periodically raid ant nests of the slave species. In these circumstances *Polyergus* soldiers aggregate in an army of 300-1000 individuals, 10-40 cm wide and some meters long.
- By visual inspection of the recorded videotapes, data about individual's distribution and density in the army have been gathered.
- Analyzing these data they found remarkable differences in the army structure in different environmental conditions such as regularity of the terrain, reciprocal visibility, etc. which may impose restrictions on the interaction range (sensitivity) among individuals.

# **Figure1:**density profile in a cross section of an army of Polyergus rufescens on different type of terrain



# Cont’d

Hence the ants during their raids seem to aggregate in a transversally organized army. Indeed, in the main direction of motion probably the dominant factor is the chemical trail produced by the scouts. So we neglect this feature since in this phase we are interested only in the aggregation phenomenon.

# features of the model

1. individuals interact directly, not via an underlying field
2. in order to avoid overcrowding, we do not consider a target density instead we introduce repulsion by means of a kernel
3. aggregation and repulsion compete but act at different scales: repulsion at the mesoscale, aggregation at the macroscale
4. the aggregation kernel may also model some environmental characteristics; in particular the dependence on the environment mentioned above is modelled by the introduction of a parameter, representing the “visibility range” of particles.
5. particles move randomly, with a variance which may depend on the population itself (e.g. via the size or via the distribution);
6. no individual drift is included

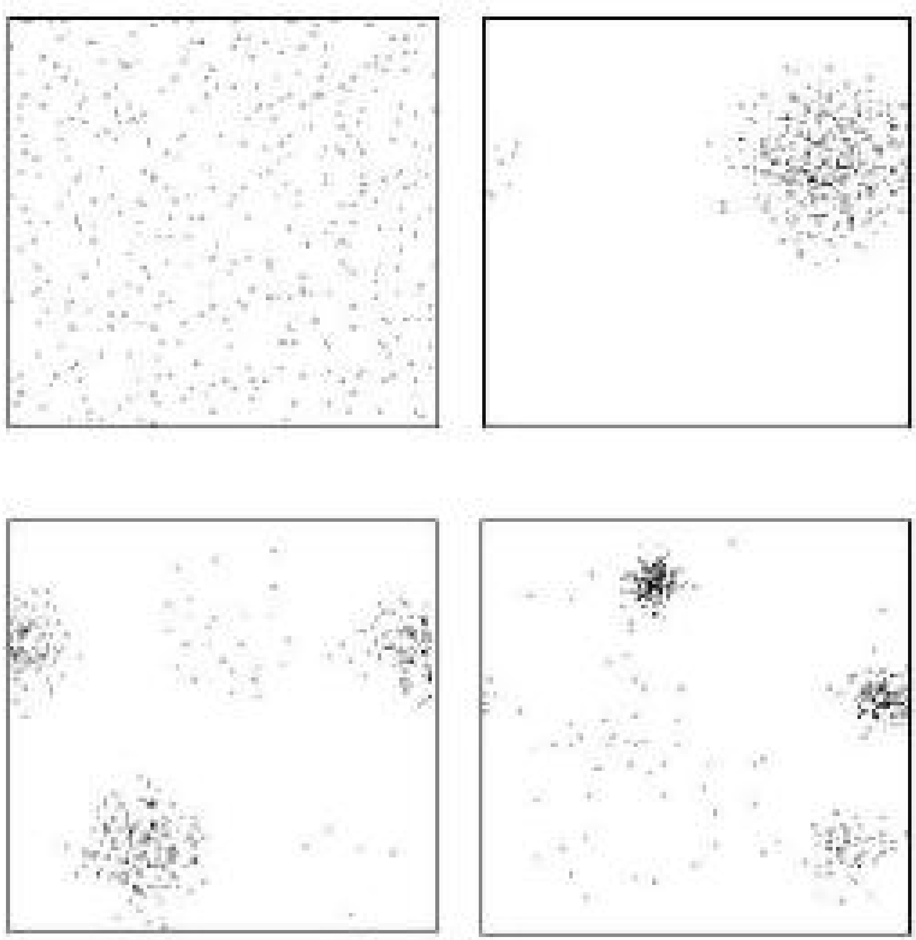
# Cont'd

**Remark**

Both kind of interaction (attraction and repulsion) are described by non local operators;

- for aggregation, the kernel is not rescaled as a function of *N*, so that the limiting equation will keep non local integral terms.
- In contrast, repulsion, being described by a moderate kernel, will be represented by local operator, in the limit: the repulsive interaction gets more and more local so that we do not expect integral terms in the limiting PDE.

**Figure2:**Cellular automata simulations carried out on a 100 X 100 lattice, with uniform initial condition, N =1000. The range of aggregation *Ra* is, 75%, 50%, 20% and 10% of the edge of the lattice.



# 2.The Lagrangian description

- The Lagrangian description of the dynamics of our system of interacting particles is given via a system of stochastic differential equations as follows:

$$dX_N^k(t) = F_N[X_N(t)](X_N^k(t))dt + \sigma_N dW^k(t) \quad k = 1,\ldots,N,$$

# Cont'd

- The drift term *F* describes the specific dynamics of the system of interacting particles, based on the modelling assumptions. The modelling assumptions are expressed by introducing in the drift term *FN* two additive components *F*1, responsible of aggregation, and *F*2, responsible of repulsion, such that *FN* = *F*1 + *F*2
- ***Aggregation***.

  Particles tend to aggregate subject to their interaction within a range of width *Ra* > 0 (finite or not).

  This corresponds to the assumption that each particle is capable of perceiving the others only within a suitable range; i.e each particle has a limited knowledge of the spatial distribution of the other particle. Aggregation is modelled by a Macroscale interaction kernel.
- A "generalized" gradient operator leads to F1 given by:

$$F_1[X_N(t)]\left(X_N^k(t)\right) = [\nabla G_a * X_N(t)]\left(X_N^k(t)\right).$$

# Cont'd

This means that each individual moves according to this generalized gradient of the measure $X_N(t)$ with respect to the kernel *Ga*; the positive sign for *F*1 expresses a force of attraction of the particle in the direction of increasing concentration of individuals.

- ***Repulsion.***

Particles are subject to repulsion when they come "too close" to each other. Repulsion is modelled by

$$F_2[X_N(t)](X_N^k(t)) = -(\nabla V_N * X_N(t))(X_N^k(t))$$
$$= -\frac{1}{N}\sum_{m=1}^{N} \nabla V_N(X_N^k(t) - X_N^m(t)).$$

# Cont'd

the negative sign for $F2$ expresses a drift towards decreasing concentration of individuals. In this case the range of the repulsion kernel decreases to zero as the size $N$ of the population increases to infinity.

Finally the system of SDE becomes:

$$dX_N^k(t) = \left( [\nabla G_a * X_N(t)](X_N^k(t)) - [\nabla V_N * X_N(t)](X_N^k(t)) \right) dt$$
$$+ \sigma_N dW^k(t) \quad k = 1, \ldots, N,$$

# Figure3 and 4: Simulation of the empirical measure

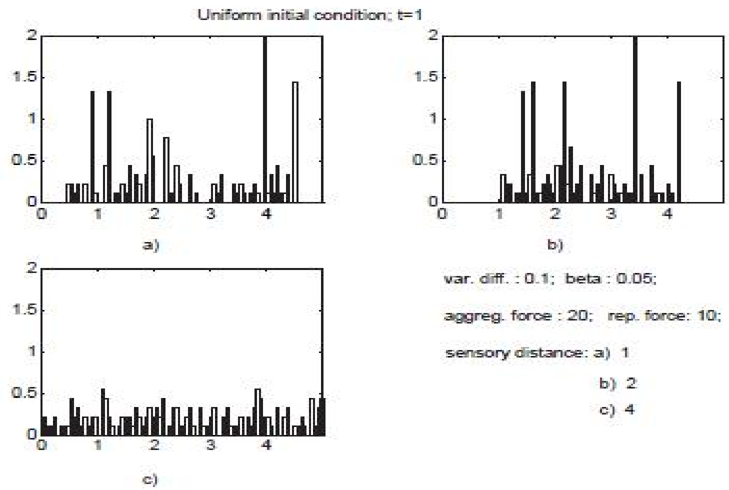



**Fig. 3.** Simulation results for the SDE model with a uniform initial distribution for different values of the range $R_a$.

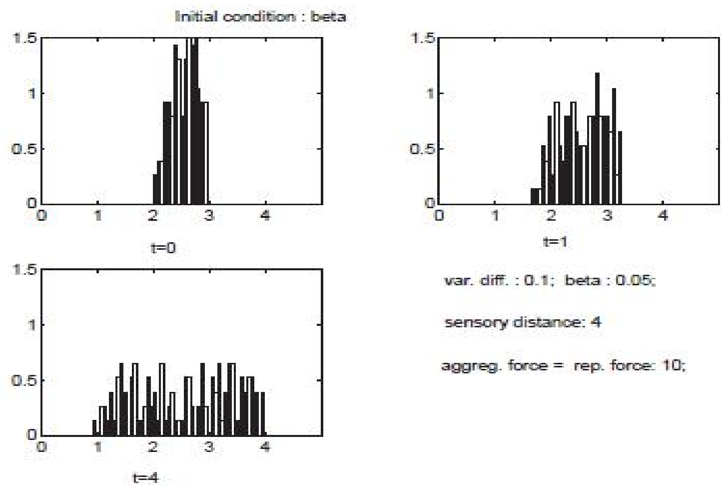



**Fig. 4.** Time evolution of the population density for the SDE model with a beta initial distribution.

# 3.Eulerian description

In order to give an Eulerian description of the system of the *N* particles, they have considered the time evolution of the empirical measure $X_N$ and used Itˆo's formula for the time evolution of any function *f* :

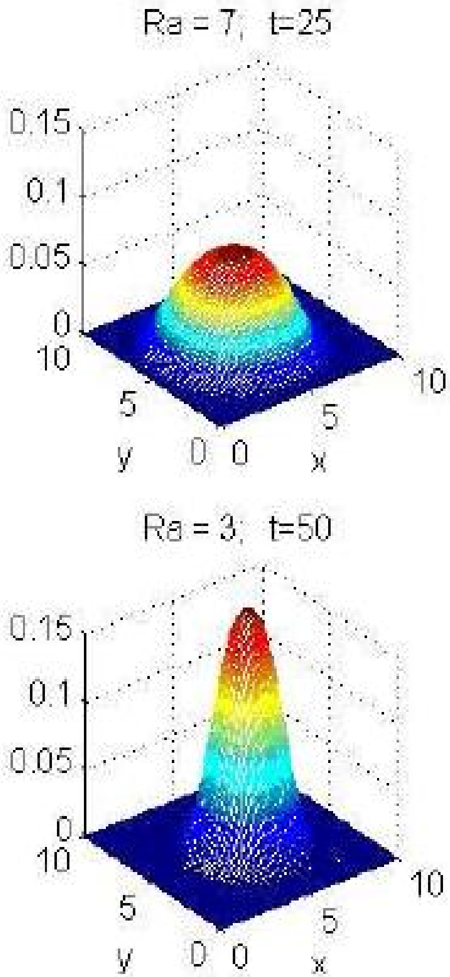
$$\begin{aligned} f(X_N^k(t),t) &= f(X_N^k(0),0) \\ &+ \int_0^t F[X_N(s)](X_N^k(s))\nabla f(X_N^k(s),s)ds \\ &+ \int_0^t \left[\frac{\partial}{\partial s} f(X_N^k(s),s) + \frac{\sigma_N^2}{2}\triangle f(X_N^k(s),s)\right] ds \\ &+ \sigma_N \int_0^t \nabla f(X_N^k(s),s)dW_s. \end{aligned}$$

# Introducing the notation

$$\langle \mu, f \rangle = \int f(x)\mu(dx),$$

$$\begin{aligned}
\langle X_N(t), f(\cdot,t)\rangle &= \frac{1}{N}\sum_{k=1}^{N} f(X_N^k(t),t) \\
&= \langle X_N(0), f(\cdot,0)\rangle \\
&\quad + \int_0^t \langle X_N(t), [(X_N(s) * \nabla G_a)(\cdot) - (X_N(s) * \nabla V_N)(\cdot)] \cdot \nabla f(\cdot,s)\rangle ds \\
&\quad + \int_0^t \left\langle X_N(t), \frac{\sigma_N^2}{2}\Delta f(\cdot,s) + \frac{\partial}{\partial s} f(\cdot,s)\right\rangle ds \\
&\quad + \frac{\sigma_N}{N}\int_0^t \sum_k \nabla f(X_N^k(s),s)dW^k(s), \qquad f \in C_b^{2,1}(\mathbb{R}^d \times [0,\infty)).
\end{aligned}$$

# Figure 5:Simulation results of the empirical measure $X_N(t)$



# **4.** Asymptotic behavior of the system for large populations

The last Equation shows how, when the number of particles *N* is large but still finite, also from the Eulerian point of view the system keeps the stochasticity which characterizes each individual.

This is not true anymore when the size of the system tends to infinity. The main reason is that the martingale term vanishes in probability.

This is the substantial reason of the deterministic limiting behavior of the process, as *N tends to infinity*, since in this limit the evolution equation of the process will not contain the Brownian noise anymore.

# Cont'd

- Supposing that the empirical process $X_N(t)$ tends as N tends to infinity to a deterministic process $X(t)$, and that it admits for any $t$, a density *rho*($x; t$) with respect to the Lebesgue measure they have showed that:

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\sigma_\infty^2}{2}\Delta\rho(x,t) + \nabla\cdot(\rho(x,t)\nabla\rho(x,t))$$

$$-\nabla\cdot[\rho(x,t)(\nabla G_a * \rho(\cdot,t))(x)], \quad x\in\mathbb{R}^d, t\geq 0,$$

$$\rho(x,0) = \rho_0(x), \quad x\in\mathbb{R}^d.$$

- If sigma infinity is greater than zero the density is smoothed by the diffusive term. This is due to the memory of the fluctuations existing when the number of particle *N* is finite. This means also that the dynamics of a single particle is still stochastic.
- If it is zero all stochasticity disappears and this brings to a degenerate equation

$$\frac{\partial}{\partial t}\rho(x,t) = \nabla\cdot\left(\rho(x,t)\nabla\rho(x,t)\right) - \nabla\cdot\left[\rho(x,t)\left(\nabla G_a * \rho(\cdot,t)\right)(x)\right], \quad x\in\mathbb{R}^d, t\geq 0.$$

*MURAKOZE*