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Dynamical Chaos Model

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Outline

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- Dynamical system
- Chaotic systems
- Some chaotic models and applications

1. Dynamical system

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- We understand by a dynamical system a set of states (called the space of states) evolving with time
- More precisely, a dynamical system is a triple (X, Φ,G). When G = Z or G = Z+ $\cup$ 0 the dynamical system is called *discrete* and it is denoted by the pair (X, f) where X is a nonempty metric space and the flow is $\Phi(n, x)$ = $fn(x)$ where f is a continuous map form X into itself.

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For $f \in C(X)$ we define its nth-*iterate* by $f^n = f \circ f^{n-1}$, $n \geq 1$,

f^0 = identity, with $f \circ g$ denoting the composition of f and g.

When $G = \mathbb{R}$ the system is called *continuous*.

The dynamical behavior is not always simple; in fact, it can be very complicated or even *chaotic*. One aspect of the chaotic nature of systems is described by the *sensitive dependence on initial conditions* which means that initial close states of the system evolve separately.

The dynamical system (X, *f*) *has sensitive dependence on initial conditions (s.d.i.c.) on* $Y \subseteq X$ *if there exists an* $r > 0$ *(independent of the points of Y) such that for each point* $x \in Y$ *and for each* $\varepsilon > 0$, *there exist* $y \in Y$ *with* $\rho(x, y) < \varepsilon$ *and* $n \geq 0$ *such that* $\rho(f^n(x), f^n(y)) \geq r$.

2. Chaotic systems

- The use of the word chaos in dynamical systems was introduced by L. Li and J. Yorke in 1975
- A dynamical system is chaotic if its dynamics is complicated in an invariant set Y ($f(Y) \subseteq Y$).

- They proved also that if there exists a periodic point of period three then there is an uncountable invariant set
- S ⊂ X where X = I = [0, 1] or R suchthat for all x, y ∈ S with x ≠ y we have:

$$\lim_{n\to\infty} \sup \left| f^n(x) - f^n(y) \right| > 0 \qquad (1)$$

$$\lim_{n\to\infty} \inf \left| f^n(x) - f^n(y) \right| = 0 \qquad (2)$$

The dynamical system (X, f) where X is a metric space, is LY2-chaotic (in the sense of Li and Yorke) if there are two different points x, y in the space (a L-Y pair) such that (1) and (2) hold.

- *The dynamical system (X, f) is chaotic on $Y \subseteq X$ if in Y are held:*

1. *f is transitive*
2. *f has (s.d.i.c.)*

3. Some chaotic models and applications

Chaotic models can be availale as:

- **The Lotka-Volterra model**, also known as the predator-prey model
- **The Nonlinear Chaotic Models** which can be used the statistical analysis of financial data.
- **Chaotic advection problems:** used to define and analyse specific vortex flow cases

The Nonlinear Chaotic Models

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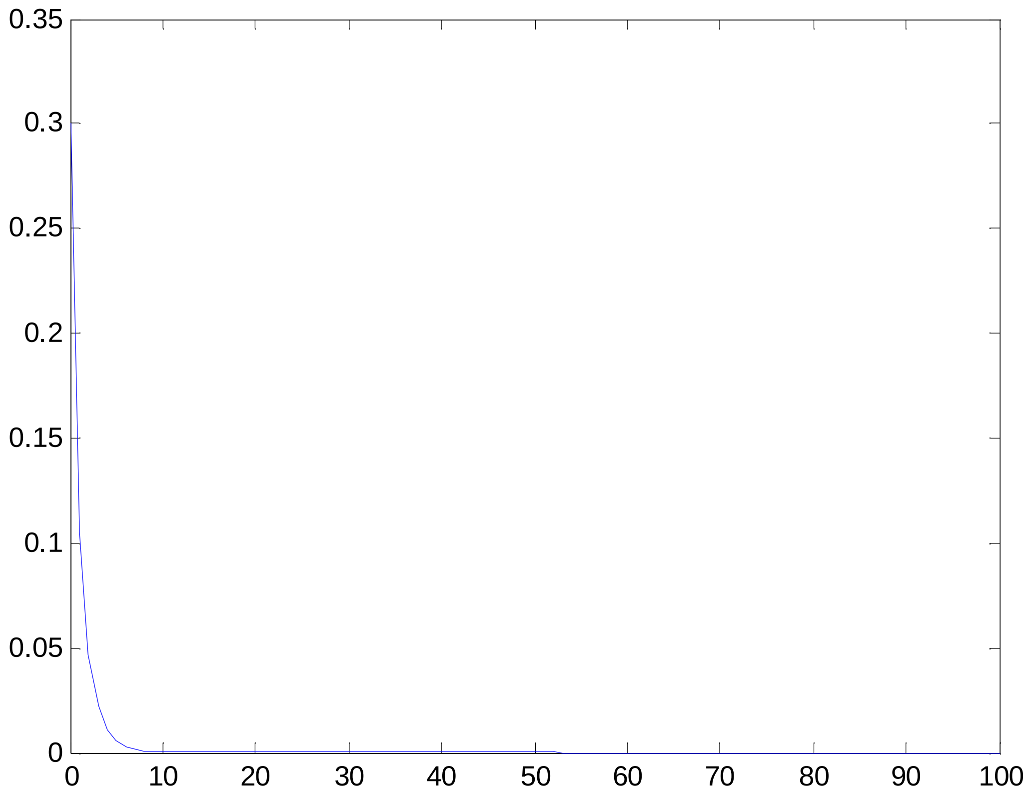
- Let s_n be the level of some 'price' at time n, and the let the sequences $h=(h_n)$ with $h_n=\ln(s_n/s_{n-1})$
- On the other hand, it is well known that even very simple nonlinear deterministic systems of the type $x_{n+1}=f(x_n;\lambda)$ or xn+1=$f(x_n,x_{n-1},\ldots x_{n-k};\ \lambda)$, where λ is a parameter, can produce sequences with behavior very similar to that of stochastic sequences
- Such systems can bring about phenomena observable in the statistical analysis of financial data

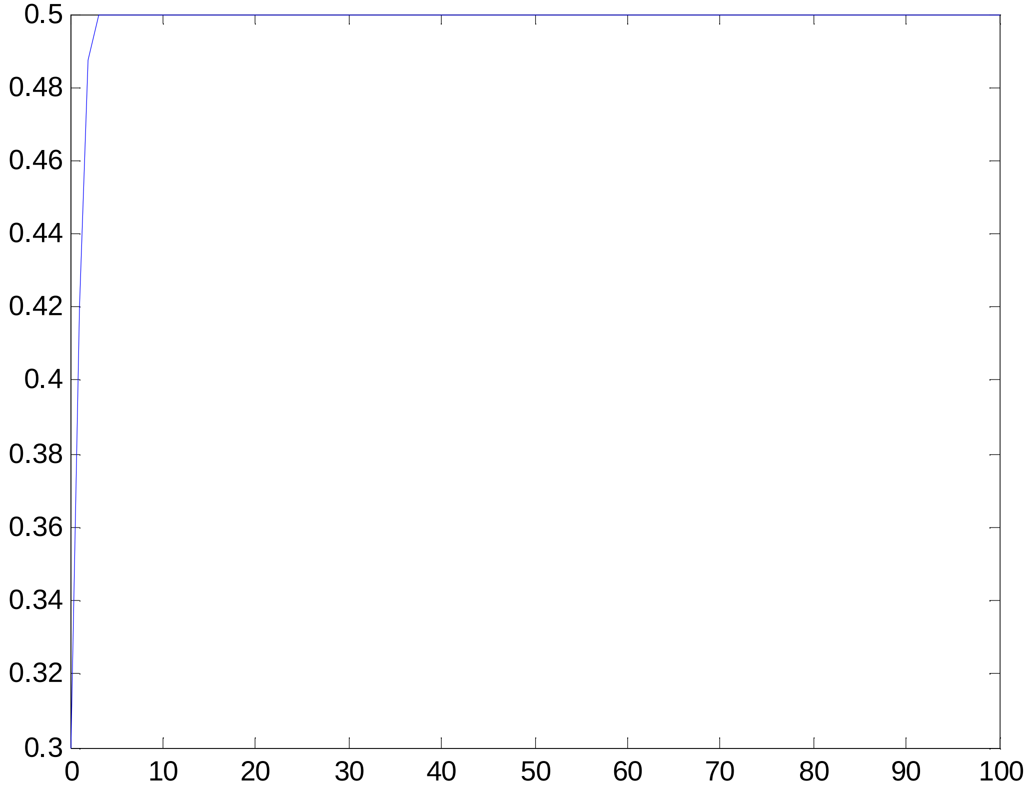
- We consider the so called the logistic map $x_{n+1}= \lambda x(1-x)$ and the corresponding one dimensional dynamical system is

$$x_{n}= \lambda x_{n-1}(1-x_{n-1}),\ n\geq 1,\ 0<x_0<1. \quad \text{(i)}$$



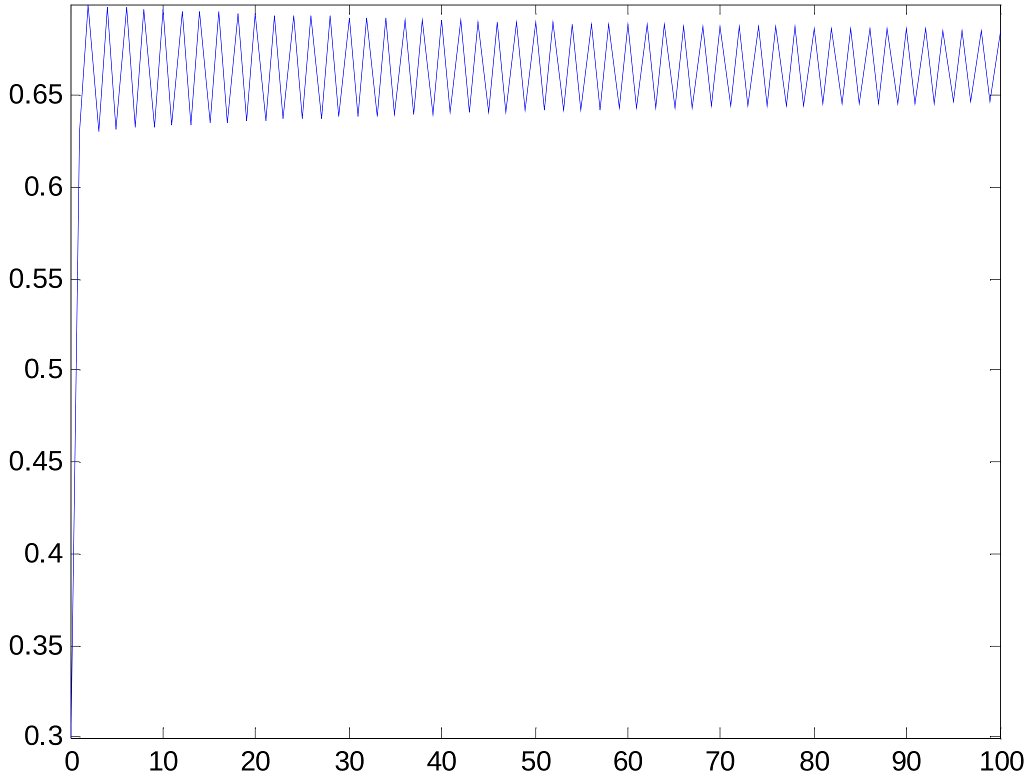
For $\lambda \leq 1$, the solution $x_n = x_n(\lambda)$ converge monotonically to 0 as $n \to \infty$ for all $0 < x_0 < 1$

Thus the state $x_\infty = 0$ is *the unique stable* state in this case, and it is the limit point of the x_n as $n \to \infty$.

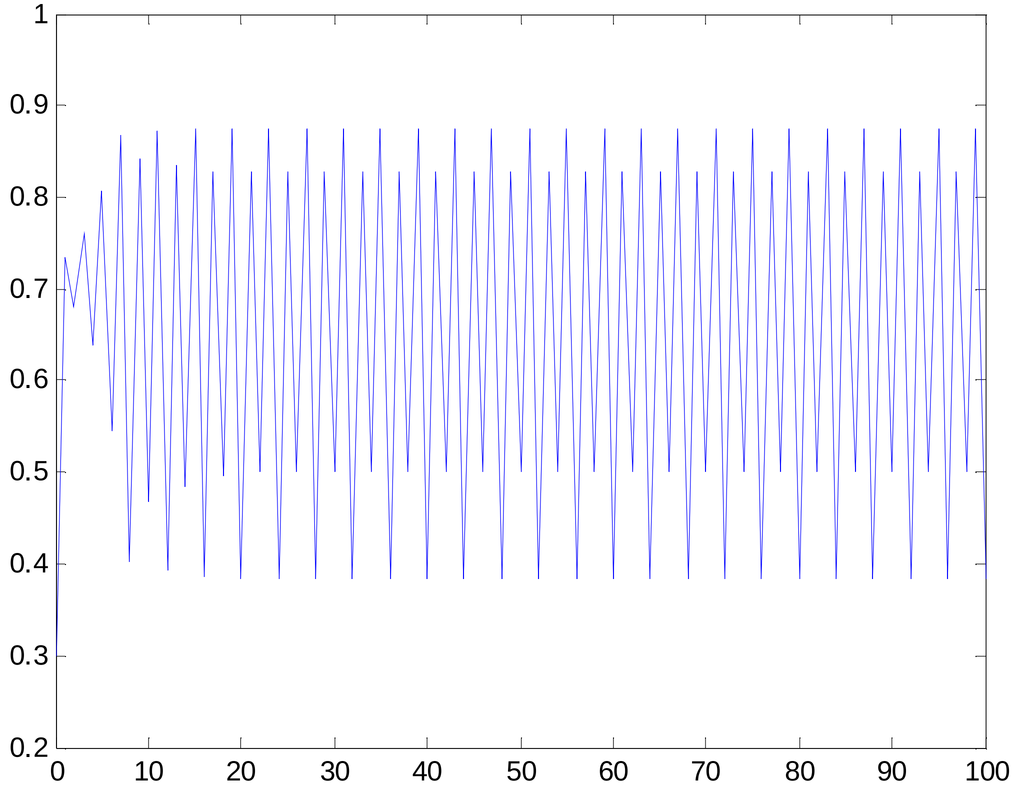


For $\lambda=2$ we have $x_n \uparrow 1/2$. Hence there also exists in this case a *unique stable* state ($x_\infty=1/2$) attracting the x_n as $n \longrightarrow \infty$.

For $\lambda<3$ the system (i) still having a unique stable state



How ever, an entirely new phenomenon occurs for $\lambda=3$: as n grows, one can distinguish two states x_∞, and the system alternates between these states.



for λ= 3.4494…: the system has now four distinguished states x_∞ and leaps from one to another

- As we increase λ there exists infinitely many such states, which is usually interpreted as a loss of stability and transition into a chaotic state
- Now the periodic character of the movements between different states is completely lost; the system wanders over an infinite set of states jumping from one to another. I t should be pointed out that, although our system is deterministic, it is impossible in practice to predict

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Thank you