LAPPEENRANTA UNIVERSITY OF TECHNOLOGY
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SIMULATING THE CURRENCY MARKETS WITH COMPUTATIONAL MARKET DYNAMICS

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ABSTRACT

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The aim of this work is to use two mathematical models to capture the dynamic nature of the currency markets and to compare the models' capability to capture those dynamics. The two models used in this work are the Jabłońska-Capasso-Morale (JCM) and the Kalman Dynamics (KD) models. Both models have been applied to EUR/USD currency pair exchange rate time series dating from 6th September 2004 to 16th March 2012. The simulation results proved that both models succeeded to capture the statistical properties of the real market time series. When comparing the two models' capability, neither of the models can be chosen to be the best model since each model performed well in one data set period and failed in another one, and vice versa. This means that none of the two models performed better than the other in all the comparison tests within the data sets periods used in this work.
Acknowledgements


Jean de Dieu Niyigaba
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<td>Autocorrelation Function</td>
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<td>ANN</td>
<td>Artificial Neural Network</td>
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<td>ARIMA</td>
<td>Autoregressive Integrated Moving Average</td>
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<td>EUR</td>
<td>Euro</td>
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<td>GARCH</td>
<td>Generalized Autoregressive Conditional Heteroscedastic</td>
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<td>GBP</td>
<td>British Pound</td>
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<td>GMM</td>
<td>Gaussian Mixture Model</td>
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<td>NNARX</td>
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<td>RPS</td>
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<td>SDE</td>
<td>Stochastic Differential Equation</td>
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<td>SGD</td>
<td>Singaporean dollar</td>
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<td>USA</td>
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1 INTRODUCTION

The foreign exchange market, the largest and most liquid financial market in the world [11], is nothing else but trading money. People would want to trade money for currency conversion or just to make money. Following the huge amount of money traded in this market, the behavior of prices in this market can help in many economic and financial decision making situations. When trading, one should understand the behavior of the rates well in advance in order to be on the right side of the move when rates will change. Therefore, the long-run behavior of the rates must be well modeled in order to understand, explain and predict the declines and increases of the rates.

Numerous studies have attempted to employ financial modeling techniques to analyze foreign exchange rates. Several models have been presented to model the prices features like jumps and spikes but failed to predict their presence accurately. Other models presented, did not take into account some important factors which may cause the rates to change. In this work we try to capture all those features for the EUR/USD currency pair exchange rates dating from 6th September 2004 to 16th March 2012. Two mathematical models will be used; these are the Jabłońska-Capasso-Morale (JCM) and the Kalman Dynamics (KD) models. These models suggests that, the psychological reaction of traders in a market may have a great influence on the exchange rates.

The work is organized as follows. The next section gives an overview of the foreign exchange market including the history, market participants, how the market works and a short review on some previous studies done on this market are presented. In section 3, a short introduction to stochastic differential equations is given and the two models are described as well as some statistical tools that will be used to compare the real and simulated prices. A description of data used in this work is given in section 4. Section 5 provides the results from the two models and a comparison of model performance is made. A brief summary and discussion are provided in section 6, and finally in section 7 some conclusions are presented.
2 FOREIGN EXCHANGE MARKET

2.1 Currency market description and history

The currency market also known as the foreign exchange market or forex market is an international market where the currency of every country is sold and bought freely [11]. Therefore, currency exchange is the simultaneous buying of one currency and selling of another. Currency exchange is necessary in many situations like payments of import and export purchases or the sale of goods or services between countries. In the forex Market there is no external control, and competition is free due to the fact that in this market the competitors can decide to transact or not. This makes the market perfect in the sense that it can not be controlled by any of the participants [11].

The foreign exchange market is considered to be the largest financial market in the world because of the great volume of transactions executed everyday, which goes up to US $4.5 trillions a day according to various assessments made. This high volume of transactions prevent any participant to take control of the market, because if one would like to do so, by changing prices at will, he would have to operate with billions of dollars [11]. The foreign exchange market spans the globe, with prices moving and currencies traded somewhere every hour of every business day [22].

Every forex trade involves the simultaneous buying of one currency and selling of another currency. These two currencies are always referred to as the currency pair in a trade. If one buys a currency pair, he buys the base currency and sells the quote currency. The base currency is the first currency in a currency pair and the quote currency is the second currency in a currency pair. The currency pair shows how much of the quote currency is needed to purchase one unit of the base currency. The bid price is refereed to as how much of the quote currency one is willing to pay to obtain a unit of the base currency, while the ask price represents how much one will obtain in the quote currency for selling one unit of the base currency. The difference between the bid price and the ask price is refereed to as the bid/ask spread [4].
2.1.1 Market history

The original beginning of the foreign exchange market is considered to be the establishment of the Bretton-Woods Accord in 1944. This accord created the concept of trading currencies against each other and originated the International Monetary Fund (IMF). Bretton-Woods Accord was established in 1944 by all the allied nations in Bretton-Woods, New Hampshire, USA, to stabilize the global economy after the World War II [22]. Between 1944 and 1971, exchange rates for foreign currencies were set at a price fixed against the US dollar which, in turn, was fixed to gold prices in hopes of bringing stability to global forex events.

On the other hand, the foreign exchange market is considered to have begun to float freely and be traded extensively in the early 1970s, after the collapse of the Bretton-Woods Accord. With the complete collapse of Bretton-Woods in 1973, world’s currencies truly began to float much more freely, and this is considered to be the beginning of the modern forex market as it is known today [8, 24].

2.1.2 Market players

More generally, there are two types of traders in the currency markets: speculative traders and consumer traders. A speculative trader is only concerned with the daily price movement, as that is where the profit potential is. A consumer trader wants a long-term ownership and is not as concerned with the daily price movements [22]. From the perspective of speculative trading, major banks have the main influence on the market since the are responsible for moving huge amounts of money. Even though these banks also offer a lot of nonspeculative currency exchanges for clients, the speculative activity is what really moves the currency markets in the most dramatic manner. Hedge funds and other investment firms also have a significant influence on the market because of the sheer enormity of the funds they dispose. Other participants in the foreign exchange market we can list are central banks, companies, and individual retail traders. These are generally not considered to have a great influence on the market [8].
2 FOREIGN EXCHANGE MARKET

2.2 Currency market functioning

Currency market has no specific location where it operates. It is generally not possible to go to a specific building and see the market where prices of foreign exchange are determined. In some cases, foreign exchange business is done over the telephone between specialists of major banks. Dealers specialized in one or more number of markets usually operate in a room with several telephones and surrounded with video screens. Following this lack of physical location, trading is conducted 24 hours a day, 5 days a week and this is an advantage for traders because they can choose when it is most convenient for them to trade, considering their own personal schedules. Currencies are executed in pairs, for example, the Euro and the US dollar (EUR/USD) or the British pound and the Japanese yen (GBP/JPY) [9, 11].

More generally, the currency market consists of two tiers: the interbank or wholesale market and the retail or client market. In the interbank market, individual transactions usually involve large sums that are multiple of million USD or the equivalent value in other currencies. Transactions are executed on a spot, forward, or swap basis. A spot transaction in the interbank market is the purchase of foreign currency, with delivery and payment between banks to take place, normally, on the following business day. A forward transaction requires delivery of foreign currency at some future date. A swap transaction in the inter-bank market is the simultaneous purchase and sale of a given amount of foreign exchange for two different value dates. In the retail market, transactions usually involve smaller amounts of money [15].

2.3 Market past modelling efforts – literature review

A number of researchers have been attracted by the currency market. This section presents some of the major studies that have been conducted on the currency market.

Paolo Pasquariello in his study [25] introduced a structural model to analyze the intra-day relationship between bid-ask spreads and market return volatility for quotes posted in a truly global around-the-clock market setting. It was done for the US dollar/Deutschemark exchange rate in 1996. Several parameters were treated via GMM (Gaussian Mixture Model) using a set of convenient
unconditional intra-day moments implying from the basic configuration of the model.

Also, Peter Mikkelsen in his study [23] developed a cross currency LIBOR Market Model for pricing options to model local term structures of currencies.

In the study by Dontwi et al. [12], a GARCH model was applied in order to find the most suitable approach to the valuation of foreign currency options in an underdeveloped financial market. To illustrate the model, the Ghanaian financial market was used in that study. The analysis included the fitting of the Generalized Autoregressive Conditional Heteroskedastic (GARCH) specification for the Ghana Cedi (GH) and European Euro (EUR) exchange rate and the implementation of the gained volatility measure in the pricing of Forex options in the local market.

In [10], a Multi-Agent Based Model was used to simulate the behaviour of the foreign exchange market. This model enables the replication of the market price and the study of the complex structure of the foreign exchange market. The model consisted of two layers so that it could replicate both the inter-dealer and the retail components of the foreign currency exchange market. Extensive experiments were made to describe and validate the model. The model was analyzed through a replication test against the EUR/USD asset pair and was able to replicate the foreign market with a good performance over a period of 5 years. This replication was reached by controlling the ratio between the buyer and the seller agents using a Bayesian network.

John Dukich, Kyung Yong Kim and Huan-Hsun Lin [13] applied the GARCH model to the logarithmic exchange rates of three exchanges rates sequences, the british pound GBP/US dollar, the Japanese YEN/US dollar and the EURO/US dollar. For each logarithmic exchange rates sequence, three GARCH models was fitted with an attempt to replicate the exchange rates via simulation. The results proved that Garch models does not accurately reflect the empirical nature of the logarithmic exchanges rates.

In [1], Neural Network Autoregressive with Exogenous Input (NNARX), Artificial Neural Network (ANN), GARCH and Autoregressive Integrated Moving Average (ARIMA) models were applied to the singaporean dollar over US dollar (SGD/USD) exchange rates in three time horizons. Only the NNARX succeeded to predict the exchange rates.
In the study of Joarder Kamruzzaman and Ruhul A. Sarker [20], three Artificial Neural Network (ANN) based forecasting models were applied to predict six different currencies against the Australian dollar, and also were compared for performance against the ARIMA model. To build the models, five moving average technical indicators were used. The results showed that all the ANN models outperform the ARIMA model and can closely forecast the exchange rates.

The artificial Neural network models were also used in [2] to predict the monthly average exchange rates of Bangladesh, and they were tested for performance against ARIMA model. The performance of the models has been evaluated using the mean absolute error (MAE), root mean squared error (RMSE) and mean absolute percentage error (MAPE). The results have shown that ANN model can predict the exchanges rates better than the ARIMA model.

Numerous other scientific models like the ones discussed above have been applied to the currency markets to try to better understand the dynamics of the prices. However, they did not take into account some factors that might have a big influence on the prices. The models proposed in this work suggest that the psychological reactions of the currency market participants may have a big influence on the exchange rate prices.

3 MODELLING BACKGROUND

3.1 Introduction to stochastic differential equations (SDEs) and stochastic modelling

Mathematical models based on ordinary differential equations (ODEs) have had a great effect on the development of modern science. They provide means of quantifying the behaviour of physical systems, and thus predictions can be made about them. However, ODEs failed to describe some systems, like small-scale systems in an adequate way, because these systems are usually subjected to random effects that are difficult to model deterministically.

The failure to capture the noisy behaviour of some systems led to the introduction of stochastic differential equations (SDEs). An ODE gives a recipe
for constructing a function by specifying how the function evolves over time. A stochastic differential equation does the same thing, but adds some form of noise to the evolution. Hence, we can define a stochastic differential equation as an equation that contains one or more stochastic terms, where the noise term has been incorporated to model uncertainties [21, 17].

Consider a stochastic process $X$ on a probability space $(\Omega, F, P)$. The Ito integral of a process $X$ from 0 to $T$ with respect to a Brownian motion $W$ is defined as

$$\int_0^T X_u dW_u = \lim_{\Delta t \to 0} \sum_{n=0}^N X_{n\Delta t}(W_{(n+1)\Delta t} - W_{n\Delta t}). \quad (1)$$

The differential $dW_u$ of Brownian motion $W_u$ is called white noise.

As we have seen above, an SDE is an ODE in which a form of noise was added to the evolution. Informally, it can be written as

$$\frac{dx}{dt} = a(X_t) + \text{Noise} \quad (2)$$

Just as a first-order ordinary differential equation can be written in the form

$$X_t = X_0 + \int_{t_0}^t a(X_u)du. \quad (3)$$

Equation 3 can be written in integral form as follows

$$X_t = X_0 + \int_{t_0}^t a(X_u)du + \int_{t_0}^t b(X_u)dW_u. \quad (4)$$

The last integral is the Ito integral and it has been used as a model of noise. The stochastic integral differential equation can also be written as follows

$$dX = a(X_t)dt + b(X_t)dW_t \quad (5)$$

### 3.2 Jabłońska-Capasso-Morale (JCM) model

Many researchers have sought to offer a psychological explanation for the financial markets behavior. This psychological explanation is related to the animal spirit idea which was introduced by John Maynard Keynes in 1936 to describe emotions which influence human behavior. In the case of financial markets, this idea refers to the psychological factors that move the market, such as trust and confidence. For instance, in any animal swarm like fish school or bird flock, each individual moves with respect to a set of basic rules that do
not come from any central coordination. The animals move in a specific direction, remain close and, at the same time, avoid collisions with their neighbors, usually separated by a certain distance. But, despite the lack of central coordination, there is always a number of individuals that pull the whole group. This swarm behavior of animals is seen between traders in a market. Traders express their bids according to their neighborhood and if a sufficiently big subgroup of traders bid far enough from the others, it may cause the rest of traders to follow that group [18, 17].

Different models used in mathematical biology have been used to study traders’ behavior in financial markets. One example would be The Capasso-Bianchi system of stochastic differential equations used for modeling animal population dynamics or price herding [3, 6, 5], in which the movement of each particle $k$ in a population of $N$ individuals depends on the position of each individual with respect to the whole population $f(X^k_t)$, as well as on its local interaction with the closest neighbors $h(k, X_t)$. The following is the Capasso-Bianchi system of stochastic differential equations in general form:

$$dX^k_t(t) = [f(X^k_t) + h(k, X_t)]dt + \sigma dW^k(t) \tag{6}$$

In this model, the Wiener process increment $dW^k(t)$ helps to maintain the randomness of the whole system.

Many authors have tried to join mathematical biology with financial time series modeling. Examples would be models proposed to study price herding phenomenon. However, those models failed to produce jumps in the process which are an important characteristic of the real price [18]. Later, a combination of the mean-reverting jump diffusion model with the Capasso-Bianchi system of stochastic differential equations has been used to model traders’ prices where traders’ prices were represented as an ensemble [19].

Since it has been proved that price spikes originate in human psychology and get magnified through the market momentum, and because spikes were generated through jump processes, jump processes have been removed to make spikes occur based on price dynamics alone. Also, the global interaction as Burgers’ type momentum component $h(k, X_t)$ was introduced. The extended model is shown in equation 7.

$$dX^k_t = [\gamma_t(X^*_t - X^k_t) + \theta_t(h(k, X_t) - X^k_t)]dt + \sigma_t dW^k_t \tag{7}$$

In this model, $\theta_t$ represents the strength of the global interaction at time $t$. 
The momentum effect has been taken into consideration through the Burgers’ equation. The Burgers’ equation is used in modelling of fluid dynamics, therefore, it can be used to simulate the behavior of financial market prices.

\[ u_t + \alpha uu_x + \beta u_{xx} = f(x, t) \]  

Equation 8

In equation 8, the term \( u_{xx} \) stands for the momentum. In financial markets case, it is expressed as the traders’ movement towards the most common price, and it is magnified at higher prices. Usually, the momentum effect occurs when a sufficiently big subgroup of the whole population has significantly different behavior that deviates from the total population mean [18].

The model in equation 7 has been extended by adding a new component \( g(k, X_t) \) which stands for the local interaction, to improve the model performance [18].

\[ dX_t^k = \left[ \gamma_t(X_t^* - X_t^k) + \theta_t(h(k, X_t) - X_t^k) + \xi_t(g(k, X_t) - X_t^k) \right] dt + \sigma_t dW_t^k \]  

Equation 9

This new model is known as the Jabłońska-Capasso-Morale (JCM) model and it will be used in this study.

### 3.3 Kalman Dynamics (KD) model

The behavior of financial markets motivated researchers to build models that could be used to simulate prices in these markets. The KD model is one of those models. This model aim to simulate real prices by merging information from model output with information about correlations between decisions of the market players collected in the covariance matrix of the state vector. Actually, this model is based on JCM model and the Variational Ensemble Kalman Filter algorithm. The version of this algorithm used to build the KD takes into account the nature of financial markets, where psychological reaction of traders have an influence on the prices, thus it combines information of the second order via propagating a set of particles through a nonlinear evolutionary operator [28].

The model is initialized by repeating the first \( H \) values of the real price \( N \) times, hence, we obtain the vector \( \mathbf{u} \in \mathbb{R}^{[N,1]} \), where \( N \) is the number of market participants. The initial particles \( \mathbf{s} = (s_1, \ldots, s_{N_p}) \) constitute the set of \( N_p \) replicates of the main trajectory \( \mathbf{u} \).
Kalman dynamics based algorithm

The KD algorithm is presented as in [27].

1. Initialize $x_i, i = 1, H$ with real prices for the first $H$ days, and assign $n = H + 1$,

2. Estimate a mean reversion level from simulated data via Least Squares:

$$
\begin{pmatrix}
1 & x_{n-1} \\
1 & x_{n-2} \\
. & . \\
. & . \\
1 & x_{n-H}
\end{pmatrix}
\begin{pmatrix}
p_1 \\
p_2 \\
. \\
. \\
. \\
dx_{n-1} \\
dx_{n-2} \\
. \\
. \\
. \\
dx_{n-H}
\end{pmatrix} =
\begin{pmatrix}
p_1 \\
p_2 \\
. \\
. \\
. \\
1 \\
1 \\
. \\
. \\
. \\
x_{n-1} - x_{n-2} \\
x_{n-2} - x_{n-3} \\
. \\
. \\
. \\
x_{n-H} - x_{n-H-1}
\end{pmatrix}, \quad (10)
$$

where $x_k, k = n - H, n - 1$ is simulated pure trading price, $dx_n = x_{n+1} - x_n$ is the price return. An ensemble mode calculated from the main vector $u \in \mathbb{R}^{[N,1]}$ at each iteration is considered as KD simulated price. Equivalently, in vector-matrix representation the system 10 can be rewritten in the following way:

$$Xp = dx. \quad (11)$$

We also compute the standard deviation for $H$ days assimilation window. The latter two values ($H$ days moving average and standard deviation) are the parameters required for JCM evolutionary operator.

3. Define the prior of the price increment $\delta u^p = (\delta u^p_1, \ldots, \delta u^p_N)$, where $N$ is the number of participants, from JCM model, which is employed as an evolutionary operator;

4. Propagate the particles $\delta s = (\delta s_1, \ldots, \delta s_N)$ via JCM operator, where each of the particles $\delta s_i, i = 1, N_p$ represents price increments $\delta u^p$ supplied with a small perturbation;

5. Perform lbfgs-minimization of the quadratic cost function with respect to the vector of price increments $\delta u^*$:

$$l(\delta u^*|\delta u_{obs}) = \frac{1}{2}(\delta u^* - \delta u^p)^T C_p^{-1}(\delta u^* - \delta u^p)$$

$$+ \frac{1}{2}(\delta u_{obs} - K\delta u^p)^T C_{e_o}^{-1}(\delta u_{obs} - K\delta u^p), \quad (12)$$
where components of artificial observational vector $\delta u_{\text{obs}}$ are composed of the particles $\delta s$ in the following way:

$$(\delta u_{\text{obs}})_i = \sum_{j=1}^{N_p} \omega_{ij} \delta s_{ij}, \quad i = 1, N. \quad (14)$$

Here $\delta s_{ij}$ denotes $j$-th component of the $i$-th vector-particle $\delta s_i$, coefficients $\omega_{ij}$ are specified as Gaussian weights $\omega_{ij} = f(s_{ij} | u_i, \sigma)$, standard deviation $\sigma$ is a model parameter, $f$ stands for the normal distribution probability density function; observational operator $K$ is an identity matrix, the inverse of observational vector error covariance $C_{\varepsilon u}^{-1} = c_o I_N$ is represented by a diagonal matrix with scalar model parameter $c_o$ as a diagonal entity.

The inverse of the prior state vector $\delta u^*$ covariance is obtained from Sherman-Morrison-Woodbury (SMW) formula:

$$C_p^{-1} = \left( \bar{X} \bar{X}^T + C_{\varepsilon m} \right)^{-1} = C_{\varepsilon m}^{-1} - C_{\varepsilon m}^{-1} \bar{X} \left( I + \bar{X} C_{\varepsilon m}^{-1} \bar{X}^T \right)^{-1} \bar{X}^T C_{\varepsilon m}^{-1}, \quad (15)$$

where model error covariance $C_{\varepsilon m}^{-1} = c_m I_N$ is diagonal, and $c_m$ is the KD model parameter. We calculate sample covariance $\bar{X}$ in the following way (see [28]):

$$\bar{X} = \left( (\delta s_1 - \delta u^p), (\delta s_2 - \delta u^p), \ldots, (\delta s_{N_p} - \delta u^p) \right) / \sqrt{N_p - 1}. \quad (16)$$

6. Sample new ensemble $\delta s^* \sim N(\delta u^*, C_{p}^{\text{est}})$ using low-storage representation of the covariance estimate $C_{p}^{\text{est}}$;

7. Increment the main trajectory $u^* = u + \delta u^*$ and the particles $s_i^* = s_i + \delta s_i^*, i = 1, N_p$;

8. calculate the ensemble mode $x_n$ from incremented state vector $u^*$

9. $s \rightarrow s^*, u \rightarrow u^*, n \rightarrow n + 1$ and go to step 2.

### 3.4 KD model parameters

- $(\gamma, \theta, \xi)$ – JCM evolutionary operator parameters;

- $N_p$ – number of particles employed for filtration;
• \( H \) – the length of assimilation window for computing moving average and standard deviation;

• \( c_o, c_m \) – the inverse values of observation and model error variances, respectively. These coefficients define how strongly we trust observations and model. When model error variances are small in comparison to observational error variances, decisions of the traders are affected by change expectations of surrounding agents greater than by rational reasons based on previous price dynamics.

• \( \sigma \) – observational operator parameter; the bigger \( \sigma \) is, the more deviant are expectations, taken into account by market agents.

### 3.5 Tools for model comparison

In this section, we present some tools that will be used to compare real prices and simulated prices. These are the first four distribution moments and the distribution divergence measure.

#### 3.5.1 Distribution moments

- The *mean* or the expectation is a measure of the central location of the distribution. It is given by

\[
\bar{x} = \frac{1}{N} \sum_{j=1}^{N} x_j
\]  

(17)

where \( N \) denotes the total number of observations and \( x_j \) are the observations (data points).

- The *standard deviation* is a measure of how spread out the data in a sample are, and how close individual observation points are to the mean. A zero value indicates that all observations are the same. A larger value implies that the observation points are farther from the mean value. Mathematically, it is given by

\[
\sigma = \sqrt{\frac{1}{N-1} \sum_{j=1}^{N} (x_j - \bar{x})^2}
\]  

(18)
• The skewness is a measure of symmetry of a distribution around its mean. It can be positive or negative. A negative skewness indicates that the tail on the left side of the probability density function is longer than the right side. A positive skew indicates that the tail on the right side is longer than the left side. Zero implies a symmetric distribution.

\[
S(x) = \frac{1}{N} \sum_{j=1}^{N} \left( \frac{x_j - \bar{x}}{\sigma} \right)^3
\] (19)

• The kurtosis measures the relative peakedness or flatness of a distribution.

\[
K(x) = \frac{1}{N} \sum_{j=1}^{N} \left( \frac{x_j - \bar{x}}{\sigma} \right)^4
\] (20)

For a normal distribution, kurtosis is equal to 3. The quantity \( K(x) - 3 \) is called excess kurtosis. Thus, for a normal distribution, the excess kurtosis should be zero. A distribution with positive excess kurtosis has heavy tails, this implies that a random sample from such a distribution tends to contain more extreme values. On the other hand, a distribution with negative excess kurtosis has short tails.

3.5.2 Distribution divergence

The distribution divergence is a measure of how much two distributions differ from each other in their shapes. It belongs to a family of so-called blended \( f \)-divergence measures. Mathematically it is represented by equation 21.

\[
D_{P,Q} = \frac{1}{2} \sum_{i=1}^{k} \frac{(p_i - q_i)^2}{\beta p_i + (1 - \beta) q_i}
\] (21)

In this equation, \( k \) denotes the number of bins and that number has to be the same for both histograms; \( p_i \) are the histogram values for the first data series, \( q_i \) are the histogram values for the second data series and \( \beta \) is the measure parameter.

4 EURO/US DOLLAR EXCHANGE RATE

In this study, Forex historical daily data for EUR/USD currency pair dating from 6th September 2004 to 16th March 2012 were used to understand how
the currency market behaves. The data were downloaded from the Internet (http://www.forexrate.co.uk) and come from multiple providers all over the world, most likely big interbank liquidity providers.

The time plot of the series in Figure 1 shows us the general price movement of EUR/USD currency pair over time. We can see that the mean of the series is changing with time which means that the series is not stationary. Also, it is very likely that the local upward and downward trends in the data are resulting from global economy situation. The data clearly reveals a sawtooth-like behaviour; very typical for financial data like exchange rates and stick prices or indices. This means existence of long increasing periods, followed by dramatic sharp drops often to levels well below the initial point of the upward trend.

![Figure 1: Time line plot of the original closing price.](image)

The similar behaviors of a non-stationary series can be observed in Figure 2 where the normalized histogram against the theoretical normal probability curve of the series is presented. We can see that the series do not follow a normal distribution. Here comes an important difference between the classical econometric approaches and those presented in this work. In most classical models, the prices are expected to be normally distributed, or log-normally and then the price log-returns would have normal distribution. That is because the conservative methodologies of ARIMA or Ornstein-Uhlenbeck processes take white noise as input and are only able to produce bell-shaped output processes. However, the models used in this study and nonlinear and even though
the noise considered in them is only white noise, they are able to produce leptokurtic realizations. Therefore, data normality is not a requirement for us.

Figure 2: Histogram of the original closing price.

The autocorrelation function (ACF) and partial autocorrelation function (PACF) plotted in Figures 3 (a) and (b) confirm the earlier observed nonstationarity of the data. This is visible in the slowly decreasing ACF, and the significant first lag of the PACF.

Figure 3: (a) ACF of the original closing price. (b) PACF of the original closing price.
In order to make the series stationary, the first difference of the series can be done, that is we produce the exchange rate returns. Now, like it can be observed in Figure 4, the series seems to be stationary around the mean zero. It is important to note that for any classical time series analysis series stationarity is crucial to be satisfied. However, the models used in this work have a moving mean reversion component included and, therefore, the analysis will be done on the original and not differenced data. Nevertheless, we still present the returns to have more knowledge about specificity of the exchange rate data.

Figure 4: Time line plot of the closing price after difference transformation.

Figures 5 shows the normalized histogram against the theoretical normal probability curve of the series. We can observe that the distribution of the series gets closer to a normal distribution, though still is not completely Gaussian, but as mentioned before, it does not create any restriction for the models considered in this work.

The plots of the ACF and PACF in Figure 6 (a) and (b) after the first difference of the series show that there is practically no significant serial correlation in the series, as the coefficients for most lags lie within the significance limits.

Table 1 presents the basic statistics of the series. The kurtosis coefficient is less than 3 for the original series and greater than 3 for the differenced original rate series, which means in both cases that we do not have a normal distribution but a short-tailed and heavy-tailed distribution, respectively. The skewness for the differenced series is closer to zero than for the original series. This means
that the differenced series is closer to a normal distribution. The standard
deviation for the original data shows that the observations data points are far
from the mean but for the differenced series they are close to the mean.
Table 1: Basic Statistics for closing price.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>St.Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Closing price</td>
<td>1.3474</td>
<td>0.0962</td>
<td>0.4473</td>
<td>2.6564</td>
</tr>
<tr>
<td>Closing price difference</td>
<td>-5.5542e-005</td>
<td>0.0090</td>
<td>-0.0551</td>
<td>4.6231</td>
</tr>
</tbody>
</table>

5 MODELLING RESULTS

5.1 JCM modelling results

In this subsection, we discuss the results obtained by using the JCM model for simulation of exchange rates.

Figure 7 presents results for the time line plot of the original against simulated prices. Features like jumps and spikes can be observed in both prices. Within some short periods of time, the prices increase or decrease considerably and then come back again to the normal level. Even though significant spikes can be observed in the simulated prices, the general price trajectory tries to follow the original prices. The simulation is also able to reproduce the sawtooth-like behaviour, though the simulated prominent upward spike does not find any analogy in the real data. The local variations of both series are comparable, however, it seems to be higher for the original data.

In Figure 8, we can observe from the histograms that both price series are right-skewed. The histogram of the simulated prices has a higher peak than the original one which is also reflected in a very high value of the distribution kurtosis. Both histograms are quite irregular and multimodal.

Autocorrelation functions for original and simulated prices are presented in Figure 9 and Figure 10 respectively. Both present a positive autocorrelation. Even though ACF diminishes faster for the simulated prices, we can still clearly say that the simulation reproduces data nonstationarity. This is achieved through the model formulation. That is, the JCM model refers to the series’ recent history within a predefined time horizon, and builds the next step realizations based on that period. This builds the significance in the serial autocorrelation.
The PACF for both original and simulated prices are presented in Figure 11 and Figure 12, respectively. There is no big difference between the plots. In case of both series there is a very significant serial dependence at the first lag, whereas all the other lags remain clearly insignificant.

Table 2 summarizes the basic statistics for the original prices against the simulated prices. We can observe that the mean, the standard deviation and the skewness are approximately the same. The only big difference can be observed
in the kurtosis where we have a higher value than the original one, as was visible in the higher peak in the simulated rate distribution histogram.
5 MODELLING RESULTS

Figure 11: PACF for original closing prices.

Figure 12: PACF for simulated closing prices by JCM model.

Table 2: Basic Statistics for original price vs simulated price by JCM model.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>St.Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original Closing price</td>
<td>1.3474</td>
<td>0.0962</td>
<td>0.4473</td>
<td>2.6564</td>
</tr>
<tr>
<td>Simulated Closing price</td>
<td>1.3533</td>
<td>0.0972</td>
<td>0.5212</td>
<td>5.3528</td>
</tr>
</tbody>
</table>
5 MODELLING RESULTS

5.2 KD modelling results

This subsection discusses the results obtained by using the KD model for simulation of exchange rates. Before we start the simulation, there are two distinct one-year periods chosen for the analysis.

In Figure 13, the original prices are plotted against the whole ensemble simulated prices for period 1. We can observe that the whole ensemble tries to capture the original prices. Prices increase and decrease in both prices as it can also be seen in Figure 14 which shows the evolution of distribution of the KD simulated ensemble. Notice that in both Figure 13 and Figure 14 the ensemble widens with time.

![Real Closing price (blue) and KD simulated forecasts (colored)](image)

Figure 13: Original closing prices (magenta), KD simulated closing prices (colored) for period 1.

Figures 15 and 16 show the mean and the mode of the ensemble against the original price and the best KD simulated forecasts against the original price respectively. Also, the histograms are presented. From the histograms, there is no big difference between the original price and the ensemble mean, both the histograms are a little bit right skewed and high peaks can be observed. This can be supported by the statistics in Table 3, where the means, standard deviations, skewness and kurtosis are close enough.

Figure 17 shows the evolution of distribution of KD simulated price returns...
5 MODELLING RESULTS

Figure 14: Distributions of KD simulated ensemble of forecasts versus real closing price (magenta) for period 1.

Figure 15: Original closing price (blue), ensemble mode (green) and mean (red) of KD simulated forecasts for period 1.

and the original price returns. We can observe that both the series seem to be stationary around the mean zero.

The Ranked Probability Score (RPS) is a score used to quantify the performance of a prediction system. It is defined as shown in equation (22).

\[
RPS = \sum_{k=1}^{k} (Y_k - O_k)^2 = (Y - O)^2
\]  
(22)
5 MODELLING RESULTS

Figure 16: Real closing price (blue) and the best KD simulated forecasts (green) for period 1.

Table 3: Original closing price statistics vs Ensemble averaged statistics of KD simulated forecasts for period 1.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>St.Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>orig price</td>
<td>1.2302</td>
<td>0.0351</td>
<td>0.1934</td>
<td>1.7012</td>
</tr>
<tr>
<td>KD sim price</td>
<td>1.2018</td>
<td>0.0646</td>
<td>0.2846</td>
<td>1.9722</td>
</tr>
</tbody>
</table>

where $Y_k$ and $O_k$ denote the $k^{th}$ component of the cumulative forecast and observation vectors $Y$ and $O$ respectively. For a perfect forecast, the RPS should be zero, otherwise it is positive. The Ranked Probability Skill Score (RPSS) (used in this work) bases on the RPS values. It is defined as shown in equation 23.

\[
RPSS = 1 - \frac{RPS}{RPS_c}
\]

where $RPS_c$ is the RPS for the reference forecast. For a perfect forecast, the RPSS should be 1. Negative values indicate that the forecast is less accurate than the standard forecast. In Figure 18, the skill score for the KD model is presented. In this period, the skill score oscillate around 0.2 and 0.4. With this results we can conclude that the forecast is accurate but not perfect.

In Figure 19 the original price is plotted against the whole ensemble simulated prices for period 2. In this period, significant spikes can be observed in the simulated price, but still the simulated price trajectory tries to follow the
original price and also widens with time. The same behaviours can be observed in Figure 20 where the the distributions of the simulated ensemble forecasts are plotted against the original price.

The mean, mode and the best simulated price for period 2 are plotted in Figure 21 and Figure 22, respectively. Histograms are also presented. We can see spikes in the simulated ensemble but one can say that the best price tries to follow the original price. Looking at the histograms, the original price histogram is left-skewed and the ensemble mean histogram is right-skewed. In
Figure 19: Original closing prices (magenta) KD simulated closing prices (colored) for period 2.

Figure 20: Distributions of KD simulated ensemble of forecasts versus real closing price (magenta) for period 2.

Table 4, the statistics support the above argument with a negative value of the skewness for the original price and a positive value for the simulated price. The means and standard deviations are quite similar. Another difference can be observed in the kurtosis values, where the simulated price has a higher kurtosis value than the original.

The price returns in period 2 are shown in Figure 23. They both oscillate around the mean zero, hence, seem to be stationary. The skill score for period 2 is shown in Figure 24 and it is oscillate around 0 and 0.006 which is very
5 MODELLING RESULTS

Figure 21: Real closing price (blue), ensemble mode (green) and mean (red) of KD simulated forecasts for period 2.

Figure 22: Real closing price (blue) and the best KD simulated forecasts (green) for period 2.

Table 4: Original closing price statistics vs Ensemble averaged statistics of KD simulated forecasts for period 2.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>St.Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>orig price</td>
<td>1.2807</td>
<td>0.0462</td>
<td>-0.2437</td>
<td>2.4014</td>
</tr>
<tr>
<td>KD sim price</td>
<td>1.2349</td>
<td>0.0566</td>
<td>0.3483</td>
<td>3.2392</td>
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</tbody>
</table>
close to 0. With this results we can conclude that the forecast is less accurate than the standard.

Figure 23: Evolution of distribution of KD simulated price returns and closing price returns (magenta) for period 2.

Figure 24: RPS score for closing price returns of KD model for period 2.

5.3 Comparison of model performance

In this subsection, we compare the simulation results obtained using the JCM and the KD models. The main goal here is to compare the two models’ performances. The comparison is done on two data sets, each consisting of 360 days.
Figures 25 and 26 present the time line plots for original prices against JCM and KD simulated prices within period 1 and period 2 respectively. We can observe that in period 1 the KD model has been able to follow the original price better than the JCM but in period 2 it is difficult to identify a better model.

Figure 25: Time line plot of the original vs JCM and original vs KD prices for period 1.

Figure 26: Time line plot of the original vs JCM and original vs KD prices for period 2.
In Figure 27 and 28, histograms for original prices against JCM and KD simulated prices for period 1 and period 2, respectively, are presented. In period 1, the original and simulated prices are right-skewed. In period 2, the original price is left-skewed while both the simulated prices are right-skewed. Within the two periods it is difficult to identify a better model.

Figure 27: Histogram of the original vs JCM and KD prices for period 1.

Figure 28: Histogram of the original vs JCM and KD prices for period 2.

The histogram divergence values between the original and simulated prices are shown in Figures 29 and 31 for JCM price and Figures 30 and 32 for KD price.
within the two periods. Here, the better model should be the one with lower distribution divergence value.

Figure 29: Histogram divergence of the original and JCM prices for period 1.

Figure 30: Histogram divergence of the original and KD prices for period 1.

In period 1, we can say that the JCM model is better since it has a lower distribution divergence value 14 than the KD with 37 value. In period 2, the KD has been better than the JCM with 29 value against 57 for the JCM model.

In Figures 33 and 34 that shows the ACF for periods 1 and 2 respectively, similar behaviour can be observed between the ACF of the original and JCM
prices in period 1, where they diminish faster compared to the KD price. In period 2, both the KD and the original prices are similar compared to the JCM price which is decreasing faster.

The PACF for both period 1 and 2 are presented in Figure 35 and Figure 36. At the first lag, the PACF is the same for both prices. Within the first period, the PACF for the JCM price is very similar to the original and lies within the confidence interval while for the KD some lies outside the interval. In period

Figure 31: Histogram divergence of the original and JCM prices for period 2.

Figure 32: Histogram divergence of the original and KD prices for period 2.
2, we can observe high PACF at first 3 lags for the JCM but still the JCM price is close to the original compared to the KD.

Figure 33: ACF of the original vs JCM and KD prices for period 1.

Figure 34: ACF of the original vs JCM and KD prices for period 2.

Table 5 and Table 6 present the basic statistics for the two periods. We can observe that in period 1, the values for the KD are closer to the original when compared to the JCM. In period 2, both the prices seem to get close to the original except for the kurtosis value which is a little bit high for the JCM.

The Root Mean Square Error (RMSE) which is a measure of how far are the
5 MODELLING RESULTS

Figure 35: PACF of the original vs JCM and KD prices for period 1.

Figure 36: PACF of the original vs JCM and KD prices for period 2.

Table 5: Basic Statistics for period 1.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>St.Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
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<tr>
<td>Orig price</td>
<td>1.2301</td>
<td>0.0350</td>
<td>0.1855</td>
<td>1.6901</td>
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<tr>
<td>JCM sim price</td>
<td>1.3246</td>
<td>0.1397</td>
<td>1.8576</td>
<td>5.9981</td>
</tr>
<tr>
<td>KD sim price</td>
<td>1.2578</td>
<td>0.0507</td>
<td>0.5515</td>
<td>2.1662</td>
</tr>
</tbody>
</table>
simulated prices values to the original prices values, has also been used to evaluate the models performance. Mathematically, it is calculated as shown in equation 24.

$$RMSE = \sqrt{\frac{\sum_{k=1}^{n}(x_{k}^{\text{mo}} - x_{k}^{\text{true}})^2}{n}}$$  \hspace{32pt} (24)$$

In Figure 37 and Figure 38 the RMSE values for the JCM and KD are plotted in period 1 and period 2 respectively. We can observe jumps in both periods, but in general, in both periods the KD seem to have lower error than the JCM model. In both periods, the mean and the standard deviation values of the errors for the KD are lower compared to those for the JCM model, as it can be observed in Table 7.

<table>
<thead>
<tr>
<th></th>
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<th>St.Dev</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>orig price</td>
<td>1.2807</td>
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<td>2.3956</td>
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<tr>
<td>JCM sim price</td>
<td>1.2513</td>
<td>0.0297</td>
<td>0.3061</td>
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</tr>
<tr>
<td>KD sim price</td>
<td>1.2778</td>
<td>0.0389</td>
<td>0.5628</td>
<td>2.7075</td>
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</tbody>
</table>

Figure 37: RMSE for JCM and KD simulated prices for period 1.
6 RESULTS SUMMARY AND DISCUSSION

Figure 38: RMSE for JCM and KD simulated prices for period 2.

Table 7: RMSE statistics for the KD and JCM models for period 1 and period 2.

<table>
<thead>
<tr>
<th>Period</th>
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<tr>
<td>JCM</td>
<td>0.0060</td>
<td>0.0074</td>
</tr>
<tr>
<td>KD</td>
<td>0.0022</td>
<td>0.0013</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Period</th>
<th>mean</th>
<th>st.dev</th>
</tr>
</thead>
<tbody>
<tr>
<td>JCM</td>
<td>0.0020</td>
<td>0.0015</td>
</tr>
<tr>
<td>KD</td>
<td>0.0011</td>
<td>0.0008</td>
</tr>
</tbody>
</table>

6 RESULTS SUMMARY AND DISCUSSION

This section summarizes and discusses the results obtained in this work. The results were divided into 3 parts. First, simulations were done using the JCM model. Next, we did simulations using the KD model, and last, a comparison of the two models was made.

To obtain the JCM results, the Matlab software was used. The results which was considered to be the best and was used in this work, were obtained with an ensemble size of 400. However, it is important to note that a change in the ensemble size does not influence the results in this model. The results obtained include the time line plots, histograms, ACF and PACF presented in
Figures 7 to 12, and the basic statistics presented in Table 2. All this figures and statistics compare the original price and the simulated price produced by the JCM model, and prove that the JCM was able to capture the dynamics of the exchange rates prices.

The KD results were also obtained using Matlab software. The results obtained are presented in Figure 13 to 24 and statistics are shown in Tables 3 and 4. All this results are from two distinct one-year periods that were chosen for the analysis. The results shows that the KD was also able to capture the dynamics of the exchange rates prices.

To evaluate the two models’ performance, a comparison of them was made. The comparison include comparing the time line plots, histograms, distribution divergence, ACF, PACF, and the RMSE of the simulated prices produced by each model against the original price. Two data set periods consisting of 360 days each were used in the comparison. Figures 25 to 38 and Tables 5, 6 and 7 are the results implemented for comparison. None of the two models succeeded to perform better than the other in all the periods. This lead us to conclude that both the models have the same power to capture the dynamics of the exchange rates prices.

7 CONCLUSIONS

In this work, we have used the Jabłońska-Capasso-Morale (JCM) and the Kalman Dynamics (KD) models to capture the statistical features of the real exchange rate prices for the EUR/USD currency pair, and we compared their capability to capture those features. The data set used cover the period from 6th September 2004 to 16th March 2012.

Before we start the simulations, an analysis of the original data was made with aim to find patterns in the data set. From the graphical presentations and basic statistics, the series was identified non-stationary. However, since both the models used in this study use the concept of moving mean reversion, either our series was stationary or not, had non influence on the rest of the study.

Simulations were done using each model separately and results were obtained.
For both models, the results proved that they both have been able to capture the behaviours of the real exchange rate prices with the same ability, hence none of the two models have been chosen to be the best compared to the other. Also the results prove that the suggested hypothesis that the psychological factors may drive the currency markets may be true.
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