Fluid Dynamics and Kalman filtering for CMD

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Reynolds averaged Navier-Stokes equations (RANS)

Consider a system of Navier-Stokes equations:

$$\rho\left(\frac{\partial u}{\partial t} + u\nabla u\right) = -\nabla p + \mu\nabla^2 u \tag{1}$$

We divide the state into mean flow and small perturbations: $u = \bar{u} + u'$, $p = \bar{p} + p'$, and obtain a system for the averaged quantities:

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\nabla\bar{u}\right) = -\nabla\bar{p} + \mu\nabla^2\bar{u} + E \tag{2}$$

where $\bar{u}$ is mean flow and E contains everything that depends on u'. Equation 2 is used for numerical simulations in fluid dynamics for modeling turbulence, and the term E goes by the name Reynolds stresses.

The pertubation dependent term E

$$E_i = -\rho \sum_{j=1}^{n} \nabla_j \underbrace{\overline{u_i' u_j'}}_{\substack{\text{the mean} \\ \text{of correlations} \\ \text{across dimensions}}} \tag{3}$$

$\bar{u}$ and u' can be also treated as a mode and an ensemble, correspondingly. From this Bayesian, or semi-Lagrangian, perspective

$$E = -\rho \nabla \left((u - \bar{u})(u - \bar{u})^T \right)$$

is the covariance around the mode that plays the role of an external force: a momentum sink.
In this case, VEnKF can be thought of as a semi-Lagrangian particle scheme to compute advection in Hilbert space.

Macro-, meso- and micro-scale dynamics in CMD

- Pressure is a macro-scale potential that is a function of particle density. It represents asset price as a function of demand at a certain trading position
- Particle density represents demand at a certain trading position
- Price as a vertical axis therefore corresponds to an isobaric pressure coordinate
- Price histogram corresponds to a "vertical" discretization - pressure is a scalar field
- Velocity u is supported on an unknown manifold, but we see only a pressure histogram covariance "field".
- We can sample this covariance field at various numerical and temporal resolutions

Macro-, meso- and micro-scale dynamics in CMD

- If we can identify a "persistent homology" from such sampling, that can be argued to represent the topology of the "domain of Animal Spirits".
- If we assume some form of ergodicity, such temporal topology could then be re-interpreted as spatial topology, as in MCMC
- But in every case, we can interpret the perturbation u' as a sample from an infinite dimensional manifold
- This sample is numerically discretized on the isobaric price coordinate by choosing a histogram division.

Macro-, meso- and micro-scale dynamics in CMD

- By Kalman Dynamics, we estimate the state of the fluid locally around our "pressure gauge", i.e. the price axis
- The mode of pressure becomes our new (Bayesian) posterior, or $\bar{p}$, of the system price
- The macro-scale dynamics are the mean flow dynamics
- The micro-scale dynamics are the locally linearized dynamics of the flow perturbation covariances $E_i = -\rho \sum_{j=1}^{n} \nabla_j (\overline{u_i' u_j'})$, computed by a PTM, i.e. VEnKF

Macro-, meso- and micro-scale dynamics in CMD

- The meso-scale coupling between the two are obtained in the micro $->$ macro direction by a diagnostic equation for pressure that decribes value formation from demand, e.g. by:

$$\rho_t = \nabla \cdot u$$

$$p = p(\rho)$$

- The macro $->$ micro direction is obtained by the pressure term in the momentum equation for the perturbation u':

$$\rho \left(\frac{\partial u'}{\partial t} + \bar{u} \nabla u' \right) = -\nabla \bar{p} + \mu \nabla^2 \bar{u} + E \tag{4}$$

Macro-, meso- and micro-scale dynamics in CMD

- Altogether this scheme can be seen as a semi-Lagragian leap-frog scheme for Navier-Stokes equations in infinite dimensions
- From the perspective of Kalman Dynamics, mean flow quantities $(\bar{u}, \bar{p})$ represent the prior state
- Perturbed quantities (u', p') are used to construct the prior for the covariance $\bar{E}$
- The posterior of the state is obtained by one more integration of the Navier-Stokes equation

$$\rho \left(\frac{\partial \bar{u}}{\partial t} + \bar{u} \nabla \bar{u} \right) = -\nabla \bar{p} + \mu \nabla^2 \bar{u} + \bar{E} \tag{5}$$

- And the posterior of the covariance E by re-sampling an ensemble around the posterior
- The process can be iterated, if needed

Burgers' and KPZ

- In the case of a single spatial dimension x the above reasoning turns into a coupling between the Burgers' and KPZ equations
- Burgers' equation

$$u_t = u_{xx} + 2uu_x + \xi_x \tag{6}$$

is representing the micro-scale dynamics and

- The Kardar-Parisi-Zhang equation

$$h_t = h_{xx} + (h_x)^2 + \xi \tag{7}$$

the meso-scale pressure, where ξ is a noise term

- The coupling micro $->$ macro is given by

$$h = \int u dx \tag{8}$$

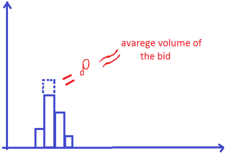
- and the macro $->$ micro by the relation

$$u = h_x \tag{9}$$

Possible treatment of pressure in terms of the market dynamics

Pressure can possibly be treated as a translation of volume of a bid. If we assume pressure to adjust immediately, one of attainable forms for pressure evolution might be

$$(\rho p)_t = \nabla \cdot u \tag{10}$$



Pressure can be also a function of density: $p = p(\rho)$, like in case of ideal gas $p = \rho^2 W$. In this case increasing density turns into increasing pressure.

Kalman Dynamics issues

- In totally isotropic situation KF alone reproduces Heat equation (i.e., the process should converge to single mean).
- Do we have a momentum effect in KF (corresponds to anizotropic situation)?
- Does KF introduce coupling between particles?

Kalman Dynamics as Jablonska-Capasso-Morale (JCM) dynamics

- The momentum term is present, and calculated in the same manner, in both dynamics
- Explicit mean reversion around a moving average is replaced by pull towards the state prior
- The herding term is replaced by the Reynolds stresses E
- The repulsive potential is replaced by the pressure term
- Brownian motion is replaced by the diffusion term

Jablonska-Capasso-Morale dynamics as Kalman Dynamics

- JCM can be seen as an approximation to Kalman Dynamics along a flow line, i.e. fully Lagrangian
- The state prior is calculated as a moving average along the mean particle path
- Reynolds stresses are localized to a fixed relative neighborhood, like in LETKF
- The pressure term is replaced by a repulsive potential
- The diffusion term is replaced by a Wiener process

Thank you!